# 变分不等式框架下结构型凸优化的分裂收缩算法

## III. 交替方向法 (ADMM) 和 PPA 意义下的 ADMM

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| 中学的数理基础 | 必要的社会实践 |
| 普通的大学数学 | 一般的优化原理 |

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# 1 Two blocks separable convex optimization

We consider the following separable convex optimization

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\ x \in \mathcal{X},\ y \in \mathcal{Y}\} \tag{1.1}$$

**Example: Best matrix approximation under some conditions**

$$\min_X \{\tfrac{1}{2}\|X - C\|_F^2 \mid X \in S^n_\Lambda \cap S_B\},$$

where

$$S^n_\Lambda = \{H \in \mathcal{S}^n \mid \lambda_{\min} I \preceq H \preceq \lambda_{\max} I\}$$

and

$$S_B = \{H \in \mathcal{S}^n \mid H_L \leq H \leq H_U\}.$$

It can be translated to the following equivalent problem:

$$\begin{array}{rl} \min_{X,Y} & \frac{1}{2}\|X - C\|^2 + \frac{1}{2}\|Y - C\|^2 \\ \text{s.t} & X - Y = 0,\ X \in S^n_\Lambda,\ Y \in S_B. \end{array} \tag{1.2}$$

The problem (1.2) is a concrete problem of type (1.1).

## Smooth Optimization Approach for Covariance Selection — Statistics

$$\min_{X} \left\{ \textbf{Tr}(CX) - \log(\det(X)) + \rho e^T |X| e \;\middle|\; X \in S_+^n \right\}$$

where $C$ is a given symmetric matrix, $e^T|X|e = \sum_{i=1}^n \sum_{j=1}^n |X_{ij}|$. Its equivalent optimization problem is

$$\begin{aligned} \min_{X,Y} \quad & \textbf{Tr}(CX) - \log(\det(X)) + \rho e^T |Y| e \\ \text{s.t} \quad & X - Y = 0, \\ & X \in S_+^n, \; Y \in R^{n\times n}. \end{aligned}$$

## Low rank and sparse optimization problem in statistics

$$\begin{aligned} \min_{X,Y} \quad & \|X\|_* + \rho e^T |Y| e \\ \text{s.t} \quad & X + Y = H \\ & X, \; Y \in R^{n\times n}. \end{aligned} \tag{1.3}$$

这些矩阵优化的数学模型本身就是一个形如 (1.1) 的结构型优化问题.

# 2 Mathematical Background

## 两大基本概念：变分不等式　和　邻近点 (PPA) 算法

**定理 1** *Let* $\mathcal{X} \subset \Re^n$ *be a closed convex set,* $\theta(x)$ *and* $f(x)$ *be convex functions and* $f(x)$ *is differentiable. Assume that the solution set of the minimization problem* $\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\}$ *is nonempty. Then,*

$$x^* \in \arg\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\} \tag{2.1a}$$

*if and only if*

$$x^* \in \mathcal{X}, \quad \theta(x) - \theta(x^*) + (x - x^*)^T \nabla f(x^*) \ge 0, \quad \forall\, x \in \mathcal{X}. \tag{2.1b}$$

## 2.1 Linearly constrained convex optimization and VI

The Lagrangian function of the problem (1.1) is

$$L^2(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b).$$

According to Lemma 1, the saddle point is a solution of the following variational inequality:

$$\left\{\begin{array}{lrl} x^* \in \mathcal{X}, & \theta_1(x) - \theta_1(x^*) + (x - x^*)^T(-A^T\lambda^*) \geq 0, & \forall\, x \in \mathcal{X}, \\ y^* \in \mathcal{Y}, & \theta_2(y) - \theta_2(y^*) + (y - y^*)^T(-B^T\lambda^*) \geq 0, & \forall\, y \in \mathcal{Y}, \\ \lambda^* \in \Re^m, & (\lambda - \lambda^*)^T(Ax^* + By^* - b) \geq 0, & \forall\, \lambda \in \Re^m. \end{array}\right.$$

Its compact form is the following variational inequality:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega, \qquad (2.2)$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(w) = \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax + By - b \end{pmatrix},$$

and

$$\theta(u) = \theta_1(x) + \theta_2(y), \qquad \Omega = \mathcal{X} \times \mathcal{Y} \times \Re^m.$$

Note that the operator $F$ is monotone, because

$$(w - \tilde{w})^T(F(w) - F(\tilde{w})) \geq 0, \text{ Here } (w - \tilde{w})^T(F(w) - F(\tilde{w})) = 0. \quad (2.3)$$

## 2.2 Preliminaries of PPA for Variational Inequalities

The optimal condition of the problem (1.1) is characterized as a mixed monotone variational inequality:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \quad (2.4)$$

**PPA for monotone mixed VI in $H$-norm**

For given $w^k$, find the proximal point $w^{k+1}$ in $H$-norm which satisfies

$$w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T \{F(w^{k+1}) + H(w^{k+1} - w^k)\} \geq 0, \ \forall\, w \in \Omega, \tag{2.5}$$

where $H$ is a symmetric positive definite matrix.

✠ Again, $w^k$ is the solution of (2.4) if and only if $w^k = w^{k+1}$ ✠

## Convergence Property of Proximal Point Algorithm in $H$-norm

$$\|w^{k+1} - w^*\|_H^2 \leq \|w^k - w^*\|_H^2 - \|w^k - w^{k+1}\|_H^2. \tag{2.6}$$

The sequence $\{w^k\}$ is Fejér monotone in $H$-norm. In customized PPA, via choosing a proper positive definite matrix $H$, the solution of the subproblem (2.5) has a closed form. An iterative algorithm is called the contraction method, if its generated sequence $\{w^k\}$ satisfies $\|w^{k+1} - w^*\|_H^2 < \|w^k - w^*\|_H^2$.

## 2.3 Augmented Lagrangian Method (ALM)

We consider the convex optimization, namely

$$\min\{\theta(u) \mid \mathcal{A}u = b, \; u \in \mathcal{U}\}. \tag{2.7}$$

The related variational inequality of the saddle point of the Lagrangian function is

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{2.8a}$$

where

$$w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \quad F(w) = \begin{pmatrix} -\mathcal{A}^T\lambda \\ \mathcal{A}u - b \end{pmatrix} \quad \text{and} \quad \Omega = \mathcal{U} \times \Re^m. \tag{2.8b}$$

## Augmented Lagrangian Method

The augmented Lagrangian function of the problem (2.7) is

$$\mathcal{L}_\beta(u, \lambda) = \theta(u) - \lambda^T(\mathcal{A}u - b) + \frac{\beta}{2}\|\mathcal{A}u - b\|^2,$$

The $k$-th iteration of the **Augmented Lagrangian Method** [13, 16] begins with a given $\lambda^k$, obtain $w^{k+1} = (u^{k+1}, \lambda^{k+1})$ via

$$\text{(ALM)} \quad \begin{cases} u^{k+1} = \arg\min\{\mathcal{L}_\beta(u, \lambda^k) \mid u \in \mathcal{U}\}, & \text{(2.9a)} \\ \lambda^{k+1} = \lambda^k - \beta(\mathcal{A}u^{k+1} - b). & \text{(2.9b)} \end{cases}$$

In (2.9), $u^{k+1}$ is only a computational result of (2.9a) from given $\lambda^k$, it is called the intermediate variable. In order to start the $k$-th iteration of ALM, we need only to have $\lambda^k$ and thus we call it as the essential variable.

The subproblem (2.9a) is a problem of mathematical form

$$\min\{\theta(u) + \tfrac{\beta}{2}\|\mathcal{A}u - p^k\|^2 \mid u \in \mathcal{U}\} \tag{2.10}$$

where $\beta > 0$ is a given scalar and $p^k = b + \frac{1}{\beta}\lambda^k$.

**Assumption**: The solution of problem (2.10) has closed-form solution or can be efficiently computed with a high precision.

The optimal condition of (2.9) can be written as $w^{k+1} \in \Omega = \mathcal{U} \times \Re^m$ and

$$\begin{cases} \theta(u) - \theta(u^{k+1}) + (u - u^{k+1})^T\{-\mathcal{A}^T\lambda^k + \beta\mathcal{A}^T(\mathcal{A}u^{k+1} - b)\} \geq 0, \ \forall u \in \mathcal{U}, \\ \qquad (\lambda - \lambda^{k+1})^T\{(\mathcal{A}u^{k+1} - b) + \frac{1}{\beta}(\lambda^{k+1} - \lambda^k)\} \geq 0, \quad \forall \lambda \in \Re^m. \end{cases}$$

The above relations can be written as

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} u - u^{k+1} \\ \lambda - \lambda^{k+1} \end{pmatrix}^T \begin{pmatrix} -\mathcal{A}^T\lambda^{k+1} \\ \mathcal{A}u^{k+1} - b \end{pmatrix} \geq (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}),$$

for all $w \in \Omega$. Using the notations in (2.8), we get the compact form

$$\begin{aligned} \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\ \geq \quad (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}), \ \forall w \in \Omega. \end{aligned} \tag{2.11}$$

Setting $w = w^*$ in (2.11), we get

$$(\lambda^{k+1} - \lambda^*)^T(\lambda^k - \lambda^{k+1}) \geq \beta\{\theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^{k+1})\}.$$

By using the monotonicity of $F$ and the optimality of $w^*$, it follows that

$$\begin{aligned} &\theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^{k+1}) \\ &= \quad \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^*) \geq 0. \end{aligned}$$

Thus, we have

$$(\lambda^{k+1} - \lambda^*)^T (\lambda^k - \lambda^{k+1}) \geq 0. \qquad (2.12)$$

By using the above inequality, we obtain

$$\begin{aligned} \|\lambda^k - \lambda^*\|^2 &= \|(\lambda^{k+1} - \lambda^*) + (\lambda^k - \lambda^{k+1})\|^2 \\ &\geq \|\lambda^{k+1} - \lambda^*\|^2 + \|\lambda^k - \lambda^{k+1}\|^2. \end{aligned}$$

It means that

$$\|\lambda^{k+1} - \lambda^*\|^2 \leq \|\lambda^k - \lambda^*\|^2 - \|\lambda^k - \lambda^{k+1}\|^2. \qquad (2.13)$$

The above inequality is the key for the convergence proof of the Augmented Lagrangian Method.

# 3 ADMM for two-block problems

Recall the separable convex optimization problem

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\ x \in \mathcal{X},\ y \in \mathcal{Y}\}.$$

The augmented Lagrangian function

$$\mathcal{L}_\beta(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b) + \frac{\beta}{2}\|Ax + By - b\|^2.$$

Applied ALM to solve the problem (1.1), the $k$-th iteration begins with given $\lambda^k$,

$$\begin{cases} (x^{k+1}, y^{k+1}) \in \arg\min\big\{\mathcal{L}_\beta(x, y, \lambda^k) \;\big|\; x \in \mathcal{X}, y \in \mathcal{Y}\big\}, & \text{(3.1a)} \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). & \text{(3.1b)} \end{cases}$$

ADMM is a relaxed ALM for the problem (1.1),the $k$-th iteration begins with given $(y^k, \lambda^k)$,

$$\begin{cases} x^{k+1} \in \arg\min\big\{\mathcal{L}_\beta(x, y^k, \lambda^k) \;\big|\; x \in \mathcal{X}\big\}, & \text{(3.2a)} \\ y^{k+1} \in \arg\min\big\{\mathcal{L}_\beta(x^{k+1}, y, \lambda^k) \;\big|\; y \in \mathcal{Y}\big\}, & \text{(3.2b)} \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). & \text{(3.2c)} \end{cases}$$

# 两个可分离目标函数问题的 ADMM 方法 [3, 5]

**Applied ADMM to the structured COP:** $(y^k, \lambda^k) \Rightarrow (y^{k+1}, \lambda^{k+1})$

First, for given $(y^k, \lambda^k)$, $x^{k+1}$ is the solution of the following problem

$$x^{k+1} \in \text{Argmin}\left\{ \begin{array}{c} \theta_1(x) - (\lambda^k)^T(Ax + By^k - b) \\ +\frac{\beta}{2}\|Ax + By^k - b\|^2 \end{array} \middle| x \in \mathcal{X} \right\} \quad \text{(3.3a)}$$

Use $\lambda^k$ and the obtained $x^{k+1}$, $y^{k+1}$ is the solution of the following problem

$$y^{k+1} \in \text{Argmin}\left\{ \begin{array}{c} \theta_2(y) - (\lambda^k)^T(Ax^{k+1} + By - b) \\ +\frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \end{array} \middle| y \in \mathcal{Y} \right\} \quad \text{(3.3b)}$$

$$\lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). \quad \text{(3.3c)}$$

**Advantages** The $x$ and $y$ sub-problems are separately solved one by one.

**Remark** Ignoring the constant term in the objective function, the sub-problems (3.22a) and (3.22b) is equivalent to

$$x^{k+1} \in \text{Argmin}\Big\{\theta_1(x) + \frac{\beta}{2}\|(Ax + By^k - b) - \frac{1}{\beta}\lambda^k\|^2 \Big| x \in \mathcal{X}\Big\} \tag{3.4a}$$

and

$$y^{k+1} \in \text{Argmin}\Big\{\theta_2(y) + \frac{\beta}{2}\|(Ax^{k+1} + By - b) - \frac{1}{\beta}\lambda^k\|^2 \Big| y \in \mathcal{Y}\Big\} \tag{3.4b}$$

respectively. Note that the equation (3.3c) can be written as

$$(\lambda - \lambda^{k+1})\{(Ax^{k+1} + By^{k+1} - b) + \frac{1}{\beta}(\lambda^{k+1} - \lambda^k)\} \geq 0, \ \forall \lambda \in \Re^m. \tag{3.4c}$$

Notice that the sub-problems (3.4a) and (3.4b) are the type of

$$x^{k+1} \in \text{Argmin}\Big\{\theta_1(x) + \frac{\beta}{2}\|Ax - p^k\|^2 \Big| x \in \mathcal{X}\Big\}$$

and

$$y^{k+1} \in \text{Argmin}\Big\{\theta_2(y) + \frac{\beta}{2}\|By - q^k\|^2 \Big| y \in \mathcal{Y}\Big\},$$

respectively. (子问题求解有困难怎么处理放在后面讲)

**Analysis** According to Theorem 1, the solution of (3.22a) and (3.22b) satisfies

$$x^{k+1} \in \mathcal{X}, \quad \theta_1(x) - \theta_1(x^{k+1}) + (x - x^{k+1})^T \\ \big\{-A^T\lambda^k + \beta A^T\big(Ax^{k+1} + By^k - b\big)\big\} \geq 0, \ \forall\, x \in \mathcal{X} \tag{3.5a}$$

and

$$y^{k+1} \in \mathcal{Y}, \quad \theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \\ \big\{-B^T\lambda^k + \beta B^T\big(Ax^{k+1} + By^{k+1} - b\big)\big\} \geq 0, \ \forall\, y \in \mathcal{Y}, \tag{3.5b}$$

respectively. Substituting $\lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b)$ (see (3.3c)) in (3.5) (eliminating $\lambda^k$ in (3.5)), we get

$$x^{k+1} \in \mathcal{X}, \quad \theta_1(x) - \theta_1(x^{k+1}) + (x - x^{k+1})^T \\ \big\{-A^T\lambda^{k+1} + \beta A^T B(y^k - y^{k+1})\big\} \geq 0, \ \forall\, x \in \mathcal{X}, \tag{3.6a}$$

and

$$y^{k+1} \in \mathcal{Y}, \quad \theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \{-B^T\lambda^{k+1}\} \geq 0, \ \forall\, y \in \mathcal{Y}. \tag{3.6b}$$

The compact form of (3.6) is $u^{k+1} = (x^{k+1}, y^{k+1}) \in \mathcal{X} \times \mathcal{Y}$ and

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \end{pmatrix}^T \left\{ \begin{pmatrix} -A^T\lambda^{k+1} \\ -B^T\lambda^{k+1} \end{pmatrix} + \beta \begin{pmatrix} A^T B \\ 0 \end{pmatrix} \big(y^k - y^{k+1}\big) \right\} \geq 0, \tag{3.7}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

By adding and subtracting the term $\beta B^T B(y^k - y^{k+1})$, we rewrite the about variational inequality in our desirable form

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \end{pmatrix}^T \left\{ \begin{pmatrix} -A^T\lambda^{k+1} \\ -B^T\lambda^{k+1} \end{pmatrix} + \beta \begin{pmatrix} A^TB \\ B^TB \end{pmatrix} (y^k - y^{k+1}) + \begin{pmatrix} 0 & 0 \\ 0 & \beta B^TB \end{pmatrix} \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} \right\} \geq 0, \ \forall (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

Combining the last inequality with (3.4c), we have the following lemma.

**引理 1** *Let* $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1}) \in \Omega$ *be generated by* (3.3) *with given* $(y^k, \lambda^k)$*, then we have*

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \\ \lambda - \lambda^{k+1} \end{pmatrix}^T \left\{ \begin{pmatrix} -A^T\lambda^{k+1} \\ -B^T\lambda^{k+1} \\ Ax^{k+1} + By^{k+1} - b \end{pmatrix} + \beta \begin{pmatrix} A^T \\ B^T \\ 0 \end{pmatrix} B(y^k - y^{k+1}) + \begin{pmatrix} 0 & 0 \\ \beta B^TB & 0 \\ 0 & \frac{1}{\beta} I_m \end{pmatrix} \begin{pmatrix} y^{k+1} - y^k \\ \lambda^{k+1} - \lambda^k \end{pmatrix} \right\} \geq 0, \ \forall w \in \Omega. \quad (3.8)$$

For convenience we use the notations

$$v = \begin{pmatrix} y \\ \lambda \end{pmatrix} \qquad \text{and} \qquad \mathcal{V}^* = \{(y^*, \lambda^*) \,|\, (x^*, y^*, \lambda^*) \in \Omega^*\}.$$

Then, we get the following lemma:

**引理 2** *Let the sequence* $\{w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})\} \in \Omega$ *be generated by* (3.3)*. Then, we have*

$$(v^{k+1} - v^*)^T H(v^k - v^{k+1}) \geq (y^k - y^{k+1})^T B^T (\lambda^k - \lambda^{k+1}), \;\forall\, w^* \in \Omega^*, \tag{3.9}$$

*where*

$$H = \begin{pmatrix} \beta B^T B & 0 \\ 0 & \frac{1}{\beta} I_m \end{pmatrix}. \tag{3.10}$$

**Proof**. Setting $w = w^*$ in (3.8), we get

$$
\begin{aligned}
&(v^{k+1}-v^*)^T H (v^k - v^{k+1}) \\
&\geq \quad \begin{pmatrix} x^{k+1}-x^* \\ y^{k+1}-y^* \end{pmatrix}^T \begin{pmatrix} A^T \\ B^T \end{pmatrix} \beta B (y^k - y^{k+1}) \\
&\qquad + \theta(u^{k+1}) - \theta(u^*) + (w^{k+1}-w^*)^T F(w^{k+1}), \; \forall\, w^* \in \Omega^*. \qquad (3.11)
\end{aligned}
$$

Observe the first part of the right hand side of (3.11),

$$
\begin{aligned}
&\begin{pmatrix} x^{k+1}-x^* \\ y^{k+1}-y^* \end{pmatrix}^T \begin{pmatrix} A^T \\ B^T \end{pmatrix} \beta B (y^k - y^{k+1}) \\
&= \quad (y^k - y^{k+1})^T B^T \beta \big(A, B\big) \begin{pmatrix} x^{k+1}-x^* \\ y^{k+1}-y^* \end{pmatrix} \\
&= \quad (y^k - y^{k+1})^T B^T \beta \big(Ax^{k+1} + By^{k+1} - (Ax^* + By^*)\big) \\
&= \quad (y^k - y^{k+1}) B^T \underline{\beta\big(Ax^{k+1} + By^{k+1} - b\big)} \\
&= \quad (y^k - y^{k+1}) B^T \underline{(\lambda^k - \lambda^{k+1})}. \qquad (3.12)
\end{aligned}
$$

To the second part, since $(w^{k+1}-w^*)^TF(w^{k+1}) = (w^{k+1}-w^*)^TF(w^*)$ and $w^*$ is the optimal solution, it follows that

$$\begin{aligned}\theta(u^{k+1}) & -\theta(u^*) + (w^{k+1}-w^*)^TF(w^{k+1}) \\ = \quad & \theta(u^{k+1})-\theta(u^*) + (w^{k+1}-w^*)^TF(w^*) \geq 0. \end{aligned} \qquad (3.13)$$

The assertion (3.11) immediately. □

**引理 3** *Let the sequence* $\{w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})\} \in \Omega$ *be generated by* (3.3)*. Then, we have*

$$(y^k - y^{k+1})^TB^T(\lambda^k - \lambda^{k+1}) \geq 0. \qquad (3.14)$$

**Proof**. Because (3.6b) is true for the $k$-th iteration and the previous iteration, we have

$$\theta_2(y) - \theta_2(y^{k+1}) + (y-y^{k+1})^T\{-B^T\lambda^{k+1}\} \geq 0, \ \forall\, y \in \mathcal{Y}, \qquad (3.15)$$

and

$$\theta_2(y) - \theta_2(y^k) + (y - y^k)^T \{-B^T \lambda^k\} \geq 0, \ \forall \, y \in \mathcal{Y}, \tag{3.16}$$

Setting $y = y^k$ in (3.15) and $y = y^{k+1}$ in (3.16), respectively, and then adding the two resulting inequalities, we get the assertion (3.14) immediately. $\square$

Substituting (3.14) in (3.9), we get

$$(v^{k+1} - v^*)^T H(v^k - v^{k+1}) \geq 0, \ \ \forall \, v^* \in \mathcal{V}^*. \tag{3.17}$$

Using the above inequality, as in the last lecture, we have the following theorem, which is the key for the proof of the convergence of ADMM.

**定理 1** *Let the sequence* $\{w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})\} \in \Omega$ *be generated by* (3.3)*. Then, we have*

$$\|v^{k+1} - v^*\|_H^2 \leq \|v^k - v^*\|_H^2 - \|v^k - v^{k+1}\|_H^2, \ \forall \, v^* \in \mathcal{V}^*. \tag{3.18}$$

# 交替方向法收敛性证明的　再阐述

交替方向法处理的是两个可分离块的凸优化问题

$$\min\{\theta_1(x)+\theta_2(y) \mid Ax+By=b,\ x\in\mathcal{X}, y\in\mathcal{Y}\}. \tag{3.19}$$

将其拉格朗日函数 $L(x,y,\lambda)=\theta_1(x)+\theta_2(y)-\lambda^T(Ax+By-b)$ 的鞍点归结为等价的变分不等式的解点：

$$w^*\in\Omega,\quad \theta(u)-\theta(u^*)+(w-w^*)^T F(w^*)\ge 0,\quad \forall\, w\in\Omega, \tag{3.20a}$$

其中

$$w=\begin{pmatrix} x\\ y\\ \lambda\end{pmatrix},\quad u=\begin{pmatrix} x\\ y\end{pmatrix},\quad F(w)=\begin{pmatrix} -A^T\lambda\\ -B^T\lambda\\ Ax+By-b\end{pmatrix},\quad \Omega=\mathcal{X}\times\mathcal{Y}\times\Re^m. \tag{3.20b}$$

ADMM 的 $k$ 步迭代从给定的核心变量 $v^k=(y^k,\lambda^k)$ 出发

$$\left\{\begin{array}{rcl} x^{k+1} &=& \arg\min\{\theta_1(x)-x^TA^T\lambda^k+\frac{\beta}{2}\|Ax+By^k-b\|^2 \mid x\in\mathcal{X}\}, \quad (3.21a)\\ y^{k+1} &=& \arg\min\{\theta_2(y)-y^TB^T\lambda^k+\frac{\beta}{2}\|Ax^{k+1}+By-b\|^2 \mid y\in\mathcal{Y}\}, \quad (3.21b)\\ \lambda^{k+1} &=& \lambda^k-\beta(Ax^{k+1}+By^{k+1}-b). \quad (3.21c)\end{array}\right.$$

根据最优性引理 1, ADMM $k$-步迭代满足

$$\begin{cases} x^{k+1}\in\mathcal{X}, & \theta_1(x)-\theta_1(x^{k+1})+(x-x^{k+1})^T\{-A^T\lambda^k+\beta A^T(Ax^{k+1}+By^k-b)\}\ge 0,\ \forall x\in\mathcal{X},\\ y^{k+1}\in\mathcal{Y}, & \theta_2(y)-\theta_2(y^{k+1})+(y-y^{k+1})^T\{-B^T\lambda^k+\beta B^T(Ax^{k+1}+By^{k+1}-b)\}\ge 0,\ \forall y\in\mathcal{Y},\\ \lambda^{k+1}\in\Re^m, & (\lambda-\lambda^{k+1})^T\{(Ax^{k+1}+By^{k+1}-b)+\frac{1}{\beta}(\lambda^{k+1}-\lambda^k)\}\ge 0,\ \forall\lambda\in\Re^m.\end{cases}$$

利用 $\lambda^{k+1}=\lambda^k-\beta(Ax^{k+1}+By^{k+1}-b)$ 上面的式子可以整理改写成

$$\begin{cases} x^{k+1}\in\mathcal{X}, & \theta_1(x)-\theta_1(x^{k+1})+(x-x^{k+1})^T\{-A^T\lambda^{k+1}+\beta A^TB(y^k-y^{k+1})\}\ge 0,\ \forall x\in\mathcal{X}, & \text{(3.22a)}\\ y^{k+1}\in\mathcal{Y}, & \theta_2(y)-\theta_2(y^{k+1})+(y-y^{k+1})^T\{-B^T\lambda^{k+1}\}\ge 0,\ \forall y\in\mathcal{Y}, & \text{(3.22b)}\\ \lambda^{k+1}\in\Re^m, & (\lambda-\lambda^{k+1})^T\{(Ax^{k+1}+By^{k+1}-b)+\frac{1}{\beta}(\lambda^{k+1}-\lambda^k)\}\ge 0,\ \forall\lambda\in\Re^m. & \text{(3.22c)}\end{cases}$$

在 (3.22b) 的后半部加上和为零的两项, 得到

$$\begin{cases} \underline{\theta_1(x)-\theta_1(x^{k+1})}+(x-x^{k+1})^T\{-A^T\lambda^{k+1}+\beta A^TB(y^k-y^{k+1})\}\ge 0,\\ \underline{\theta_2(y)-\theta_2(y^{k+1})}+(y-y^{k+1})^T\{-B^T\lambda^{k+1}+\underbrace{\beta B^TB(y^k-y^{k+1})+\beta B^TB(y^{k+1}-y^k)}\}\ge 0,\\ (\lambda-\lambda^{k+1})^T\{(Ax^{k+1}+By^{k+1}-b)+\frac{1}{\beta}(\lambda^{k+1}-\lambda^k)\}\ge 0.\end{cases}$$

利用变分不等式 (3.20), 进行合理整合, 得到

$$\underline{\theta(u)-\theta(u^{k+1})}+(w-w^{k+1})^T F(w^{k+1})$$

$$+\begin{pmatrix} x-x^{k+1}\\ y-y^{k+1}\end{pmatrix}^T \beta\begin{pmatrix} A^T\\ B^T\end{pmatrix} B(y^k-y^{k+1})+\begin{pmatrix} y-y^{k+1}\\ \lambda-\lambda^{k+1}\end{pmatrix}^T \begin{pmatrix} \beta B^T B & 0\\ 0 & \frac{1}{\beta} I_m\end{pmatrix}\begin{pmatrix} y^{k+1}-y^k\\ \lambda^{k+1}-\lambda^k\end{pmatrix}\geq 0.$$

将上式中那个任意的 $w$, 设成解点 $w^*$ 便有

$$\underline{\theta(u^*)-\theta(u^{k+1})}+(w^*-w^{k+1})^T F(w^{k+1})$$

$$+\begin{pmatrix} x^*-x^{k+1}\\ y^*-y^{k+1}\end{pmatrix}^T \beta\begin{pmatrix} A^T\\ B^T\end{pmatrix} B(y^k-y^{k+1})+\begin{pmatrix} y^*-y^{k+1}\\ \lambda^*-\lambda^{k+1}\end{pmatrix}^T \begin{pmatrix} \beta B^T B & 0\\ 0 & \frac{1}{\beta} I_m\end{pmatrix}\begin{pmatrix} y^{k+1}-y^k\\ \lambda^{k+1}-\lambda^k\end{pmatrix}\geq 0.$$

经转换, 得到

$$\begin{pmatrix} y^{k+1}-y^*\\ \lambda^{k+1}-\lambda^*\end{pmatrix}^T \begin{pmatrix} \beta B^T B & 0\\ 0 & \frac{1}{\beta} I_m\end{pmatrix}\begin{pmatrix} y^k-y^{k+1}\\ \lambda^k-\lambda^{k+1}\end{pmatrix}$$

后面记 $v=\begin{pmatrix} y\\ \lambda\end{pmatrix}$, $H=\begin{pmatrix} \beta B^T B & 0\\ 0 & \frac{1}{\beta} I_m\end{pmatrix}$

$$\geq \begin{pmatrix} x^{k+1}-x^*\\ y^{k+1}-y^*\end{pmatrix}^T \beta\begin{pmatrix} A^T\\ B^T\end{pmatrix} B(y^k-y^{k+1})+\underbrace{[\theta(u^{k+1})-\theta(u^*)+(w^{k+1}-w^*)^T F(w^{k+1})]}. \quad (3.23)$$

假如 (3.23) 式右端非负, 证明就基本上完成了. 由于

$$\theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^{k+1}) = \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^*) \geq 0.$$

(3.23) 式右端下划线部分非负. 因此从 (3.23) 式得到

$$(v^{k+1} - v^*)^T H(v^k - v^{k+1}) \;\geq\; \begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \end{pmatrix}^T \beta \begin{pmatrix} A^T \\ B^T \end{pmatrix} B(y^k - y^{k+1}). \tag{3.24}$$

对 (3.24) 式的右端进行处理, 有

$$\begin{aligned}
\begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \end{pmatrix}^T \beta \begin{pmatrix} A^T \\ B^T \end{pmatrix} B(y^k - y^{k+1}) &= (y^k - y^{k+1})^T B^T \beta (A, B) \begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \end{pmatrix} \\
&= (y^k - y^{k+1})^T B^T \beta \big(Ax^{k+1} + By^{k+1} - (Ax^* + By^*)\big) \qquad \text{利用}(Ax^* + By^* = b) \\
&= (y^k - y^{k+1})^T B^T \beta\big(Ax^{k+1} + By^{k+1} - b\big) \\
&= (y^k - y^{k+1})^T B^T (\lambda^k - \lambda^{k+1}).
\end{aligned} \tag{3.25}$$

后面我们要证明 $(y^k - y^{k+1})^T B^T (\lambda^k - \lambda^{k+1}) \geq 0.$

利用 (3.22b) 有

$$\theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T\{-B^T\lambda^{k+1}\} \ge 0, \ \forall\, y \in \mathcal{Y},$$

和 $$\theta_2(y) - \theta_2(y^k) + (y - y^k)^T\{-B^T\lambda^k\} \ge 0, \ \forall\, y \in \mathcal{Y}.$$

( 将任意的 $y$ 分别设成 $y^k$ 和 $y^{k+1}$ )

$$\theta_2(y^k) - \theta_2(y^{k+1}) + (y^k - y^{k+1})^T\{-B^T\lambda^{k+1}\} \ge 0.$$

$$\theta_2(y^{k+1}) - \theta_2(y^k) + (y^{k+1} - y^k)^T\{-B^T\lambda^k\} \ge 0.$$

(将上面两式相加, 就有) $(y^k - y^{k+1})^T B^T(\lambda^k - \lambda^{k+1}) \ge 0.$ ((3.25) 式右端非负)

证明了(3.25) 式右端非负, 进而得到 (3.24) 式右端非负. 所以

$$(v^{k+1} - v^*)^T H(v^k - v^{k+1}) \ge 0. \tag{3.26}$$

Lemma 2 告诉我们:

$$b^T H(a-b) \ge 0 \quad \Rightarrow \quad \|b\|_H^2 \le \|a\|_H^2 - \|a-b\|_H^2. \tag{3.27}$$

在 (3.27) 中置 $a = (v^k - v^*)$ 和 $b = (v^{k+1} - v^*)$, 根据 (3.26) 就得到收敛的关键不等式

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \|v^k - v^{k+1}\|_H^2.$$

由 $\|v^k - v^{k+1}\|_H^2 \le \|v^k - v^*\|_H^2 - \|v^{k+1} - v^*\|_H^2$ 得 $\sum_{k=0}^{\infty} \|v^k - v^{k+1}\|_H^2 \le \|v^0 - v^*\|_H^2.$

**How to choose the parameter $\beta$.** The efficiency of ADMM is heavily dependent on the parameter $\beta$ in (3.3). We discuss how to choose a suitable $\beta$ in the practical computation.

Note that if $\boldsymbol{\beta A^T B(y^k - y^{k+1}) = 0}$, then it follows from (3.7)

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \end{pmatrix}^T \begin{pmatrix} -A^T\lambda^{k+1} \\ -B^T\lambda^{k+1} \end{pmatrix} \geq 0, \ \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

In this case, if additionally $\boldsymbol{Ax^{k+1} + By^{k+1} - b = 0}$, then we have

$$\begin{cases} \theta_1(x) - \theta_1(x^{k+1}) + (x - x^{k+1})^T(-A^T\lambda^{k+1}) \geq 0, & \forall x \in \mathcal{X} \\ \theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T(-B^T\lambda^{k+1}) \geq 0, & \forall y \in \mathcal{Y} \\ (\lambda - \lambda^{k+1})^T(Ax^{k+1} + By^{k+1} - b) \geq 0, & \forall \lambda \in \Re^m \end{cases}$$

and consequently $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is a solution of the VI (2.2).

In other words, $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is not a solution of (2.2) because

$$\beta A^T B(y^k - y^{k+1}) \neq 0 \qquad \text{and/or} \qquad Ax^{k+1} + By^{k+1} - b \neq 0.$$

We call

$$\|\boldsymbol{\beta A^T B(y^k - y^{k+1})}\| \qquad \text{and} \qquad \|\boldsymbol{Ax^{k+1} + By^{k+1} - b}\|$$

the primal-residual and the dual-residual, respectively. It seems that we should balance the primal and the dual residuals dynamically. If

$$\mu\|\beta A^T B(y^k - y^{k+1})\| < \|Ax^{k+1} + By^{k+1} - b\| \quad \text{with a} \quad \mu > 1,$$

it means that the dual residual is too large and thus we should enlarge the parameter $\beta$ in the augmented Lagrangian function. Otherwise, we should reduce the parameter $\beta$. A simple scheme that often works well is (see, e.g., [9]):

$$\beta_{k+1} = \begin{cases} \beta_k * \tau, & \text{if } \mu\|\beta A^T B(y^k - y^{k+1})\| < \|Ax^{k+1} + By^{k+1} - b\|; \\ \beta_k/\tau, & \text{if } \|\beta A^T B(y^k - y^{k+1})\| > \mu\|Ax^{k+1} + By^{k+1} - b\|; \\ \beta_k, & \text{otherwise.} \end{cases}$$

where $\mu > 1, \tau > 1$ are parameters. Typical choices might be $\mu = 10$ and $\tau = 2$. The idea behind this penalty parameter update is to try to keep the primal and dual residual norms within a factor of $\mu$ of one another as they both converge to zero. This self adaptive adjusting rule has been used by S. Boyd's group [1] and in their Optimization Solver [6].

# 4 Linearized ADMM

The augmented Lagrangian Function of the problem (1.1) is

$$\mathcal{L}_\beta^{[2]}(x,y,\lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax+By-b) + \frac{\beta}{2}\|Ax+By-b\|^2. \quad (4.1)$$

Solving the problem (1.1) by using ADMM, the $k$-th iteration begins with given $(y^k,\lambda^k)$, it offers the new iterate $(y^{k+1},\lambda^{k+1})$ via

$$\text{(ADMM)} \quad \left\{ \begin{array}{ll} x^{k+1} = \arg\min\{\mathcal{L}_\beta^{[2]}(x,y^k,\lambda^k) \mid x\in\mathcal{X}\}, & \text{(4.2a)} \\ y^{k+1} = \arg\min\{\mathcal{L}_\beta^{[2]}(x^{k+1},y,\lambda^k) \mid y\in\mathcal{Y}\}, & \text{(4.2b)} \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1}+By^{k+1}-b). & \text{(4.2c)} \end{array} \right.$$

In optimization problem, the solution is invariant by changing the constant term in the objective function. Thus, by using the augmented Lagrangian function,

$$\begin{aligned} y^{k+1} &= \arg\min\{\mathcal{L}_\beta^{[2]}(x^{k+1},y,\lambda^k) \mid y\in\mathcal{Y}\} \\ &= \arg\min\{\theta_2(y) - y^TB^T\lambda^k + \frac{\beta}{2}\|Ax^{k+1}+By-b\|^2 \mid y\in\mathcal{Y}\}. \end{aligned}$$

Thus, by denoting $q^k = b - Ax^{k+1} + \frac{1}{\beta}\lambda^k$, the solution of (3.22b) is given by

$$\min\{\theta_2(y) + \frac{\beta}{2}\|By - q^k\|^2 \mid y \in \mathcal{Y}\}. \tag{4.3}$$

In some practical applications, because of the structure of the matrix $B$, the subproblem (4.3) is not so easy to be solved. In this case, it is necessary to use the linearized version of the ADMM.

Notice that the Taylor expansion of the quadratic term of (4.2b), namely,

$$\begin{aligned} \frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \;&=\; \frac{\beta}{2}\|B(y - y^k) + (Ax^{k+1} + By^k - b)\|^2 \\ =\quad & \frac{\beta}{2}\|B(y - y^k)\|^2 + \beta(y - y^k)^T B^T (Ax^{k+1} + By^k - b) \\ & + \frac{\beta}{2}\|Ax^{k+1} + By^k - b\|^2 \end{aligned}$$

Changing the constant term in the objective function of (4.2b) accordingly, we have

$$
\begin{aligned}
y^{k+1} &= \arg\min\{\mathcal{L}_\beta^{[2]}(x^{k+1}, y, \lambda^k) \mid y \in \mathcal{Y}\} \\
&= \arg\min\{\theta_2(y) - y^T B^T \lambda^k + \frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \mid y \in \mathcal{Y}\} \\
&= \arg\min \left\{ \begin{array}{c} \theta_2(y) - y^T B^T \lambda^k + \beta y^T B^T (Ax^{k+1} + By^k - b) \\ + \frac{\beta}{2}\|B(y - y^k)\|^2 \end{array} \middle| \, y \in \mathcal{Y} \right\}.
\end{aligned}
$$

So-called linearized version of ADMM, we remove the unfavorable term $\dfrac{\beta}{2}\|B(y-y^k)\|^2$ in the objective function, and add the term $\dfrac{s}{2}\|y - y^k\|^2$.

Strictly speaking, it should be a ”linearization” plus ”regularization” method. It can also be interpreted as:

The term $\dfrac{\beta}{2}\|B(y-y^k)\|^2$ is replaced with $\dfrac{s}{2}\|y - y^k\|^2$.

In other words, it is equivalent to adding the term

$$
\frac{1}{2}\|y - y^k\|_{D_B}^2 \qquad (\text{with } \; D_B = sI_{n_2} - \beta B^T B) \tag{4.4}
$$

to the objective function of (4.2b), we get

$$
\begin{aligned}
y^{k+1} &= \arg\min\{\mathcal{L}_\beta^{[2]}(x^{k+1}, y, \lambda^k) + \frac{1}{2}\|y - y^k\|_{D_B}^2 \mid y \in \mathcal{Y}\} \\
&= \arg\min \left\{ \begin{array}{c} \theta_2(y) - y^T B^T \lambda^k + \beta y^T B^T (Ax^{k+1} + By^k - b) \\ + \frac{s}{2}\|y - y^k\|^2 \end{array} \middle| \, y \in \mathcal{Y} \right\} \\
&= \arg\min \left\{ \theta_2(y) + \frac{s}{2}\big\|y - d^k\big\|^2 \mid y \in \mathcal{Y} \right\},
\end{aligned} \tag{4.5}
$$

where

$$d^k = y^k - \frac{1}{s} B^T \big[\beta(Ax^{k+1} + By^k - b) - \lambda^k\big].$$

By using such strategy, the sub-problems of ADMM is simplified. The linearized version of ADMM are applied successfully in scientific computing [14, 17, 18, 19]. The following analysis is based on the fact that the sub-problems (3.22a) and

$$\min\{\theta_2(y) + \frac{s}{2}\|y - d^k\|^2 \,|\, y \in \mathcal{Y}\}$$

are easy to be solved.

**Linearized ADMM.** For solving the problem (1.1), the $k$-th iteration of the linearized ADMM begins with given $v^k = (y^k, \lambda^k)$, produces the $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$

via the following procedure:

$$\begin{cases} x^{k+1} = \arg\min\{\mathcal{L}_\beta^{[2]}(x, y^k, \lambda^k) \mid x \in \mathcal{X}\}, & \text{(4.6a)} \\ y^{k+1} = \arg\min\{\mathcal{L}_\beta^{[2]}(x^{k+1}, y, \lambda^k) + \frac{1}{2}\|y - y^k\|_{D_B}^2 \mid y \in \mathcal{Y}\}, & \text{(4.6b)} \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). & \text{(4.6c)} \end{cases}$$

where $D_B$ is defined by (4.4).

First, using the optimality of the sub-problems of (4.6), we prove the following lemma as the base of convergence.

**引理 4** *Let* $\{w^k\}$ *be the sequence generated by Linearized ADMM* (4.6) *for the problem* (1.1)*. Then, we have*

$$\begin{aligned} w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) &+ (w - w^{k+1})^T F(w) \\ &+ \beta(x - x^{k+1})^T A^T (By^k - By^{k+1}) \\ \geq (y - y^{k+1})^T D_B (y^k - y^{k+1}) & \\ &+ \frac{1}{\beta}(\lambda - \lambda^{k+1})^T(\lambda^k - \lambda^{k+1}), \quad \forall w \in \Omega. \end{aligned} \tag{4.7}$$

**Proof.** For the $x$-subproblem in (4.6a), by using Lemma 1, we have

$$x^{k+1} \in \mathcal{X}, \quad \theta_1(x) - \theta_1(x^{k+1}) \\ + (x - x^{k+1})^T \{-A^T\lambda^k + \beta A^T(Ax^{k+1} + By^k - b)\} \\ \geq 0, \quad \forall\, x \in \mathcal{X}.$$

By using the multipliers update form in (4.6), $\lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b)$, the above inequality can be written as

$$x^{k+1} \in \mathcal{X}, \quad \theta_1(x) - \theta_1(x^{k+1}) \\ + (x - x^{k+1})^T \{-A^T\lambda^{k+1} + \beta A^T B(y^k - y^{k+1})\} \\ \geq 0, \quad \forall\, x \in \mathcal{X}. \tag{4.8}$$

For the $y$-subproblem in (4.6b), by using Lemma 1, we have

$$y^{k+1} \in \mathcal{Y}, \quad \theta_2(y) - \theta_2(y^{k+1}) \\ + (y - y^{k+1})^T \{-B^T\lambda^k + \beta B^T(Ax^{k+1} + By^{k+1} - b)\} \\ + (y - y^{k+1})^T D_B(y^{k+1} - y^k) \geq 0, \ \forall\, y \in \mathcal{Y}.$$

Again, by using the update form $\lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b)$, the above inequality can be written as

$$
\begin{aligned}
y^{k+1} \in \mathcal{Y}, \;\; \theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \{-B^T \lambda^{k+1}\} \\
\geq (y - y^{k+1})^T D_B (y^k - y^{k+1})\}, \; \forall\, y \in \mathcal{Y}. \qquad (4.9)
\end{aligned}
$$

Notice that the update form for the multipliers, $\lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b)$, can be written as $\lambda^{k+1} \in \Re^m$ and

$$
(\lambda - \lambda^{k+1})^T \{(Ax^{k+1} + By^{k+1} - b) + \frac{1}{\beta}(\lambda^{k+1} - \lambda^k)\} \geq 0, \; \forall\, \lambda \in \Re^m. \quad (4.10)
$$

Adding (4.8), (4.9) and (4.10), and using the notation in (2.2), we get

$$
\begin{aligned}
w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\
+ \beta(x - x^{k+1})^T A^T (By^k - By^{k+1}) \\
\geq (y - y^{k+1})^T D_B (y^k - y^{k+1}) \\
+ \frac{1}{\beta}(\lambda - \lambda^{k+1})^T (\lambda^k - \lambda^{k+1}), \quad \forall w \in \Omega. \qquad (4.11)
\end{aligned}
$$

For the term $(w - w^{k+1})^T F(w^{k+1})$ in the left side of (4.11), by using (2.3), we have

$$(w - w^{k+1})^T F(w^{k+1}) = (w - w^{k+1})^T F(w).$$

The assertion (4.7) is proved. $\square$

This lemma is the base for the convergence analysis of the linearized ADMM.

**The contractive property of the sequence $\{w^k\}$ by Linearized ADMM** (4.6)

In the following we will prove, for any $w^* \in \Omega^*$, the sequence

$$\{\|v^{k+1} - v^*\|_G + \|y^k - y^{k+1}\|_{D_B}^2\}$$

is monotonically decreasing. For this purpose， we prove some lemmas.

**引理 5** *Let $\{w^k\}$ be the sequence generated by Linearized ADMM* (4.6) *for the problem*

(1.1). *Then, we have*

$$w^{k+1} \in \Omega, \;\; \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w) + \beta \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \end{pmatrix}^T \begin{pmatrix} A^T \\ B^T \end{pmatrix} B(y^k - y^{k+1}) \geq (v - v^{k+1})^T G(v^k - v^{k+1}), \;\; \forall w \in \Omega, \tag{4.12}$$

*where* $G$ *is given by*

$$G = \begin{pmatrix} D_B + \beta B^T B & 0 \\ 0 & \frac{1}{\beta} I \end{pmatrix}. \tag{4.13}$$

*.*

**Proof**. Adding $(y - y^{k+1})^T \beta B^T B(y^k - y^{k+1})$ to the both sides of (4.7) in Lemma 4, and using the notation of the matrix $G$, we obtain (4.12) immediately and the lemma is proved. □

**引理 6** *Let* $\{w^k\}$ *be the sequence generated by Linearized ADMM* (4.6) *for the problem*

(1.1). *Then, we have*

$$(v^{k+1}-v^*)^T G(v^k-v^{k+1}) \ge (\lambda^k-\lambda^{k+1})^T B(y^k-y^{k+1}), \quad \forall w^* \in \Omega^*. \tag{4.14}$$

**Proof**. Setting the $w \in \Omega$ in (4.12) by any $w^* \in \Omega^*$, we obtain

$$\begin{aligned} &(v^{k+1} - v^*)^T G(v^k - v^{k+1}) \\ &\quad \ge \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^*) \\ &\qquad + \beta \begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \end{pmatrix}^T \begin{pmatrix} A^T \\ B^T \end{pmatrix} B(y^k - y^{k+1}). \end{aligned} \tag{4.15}$$

According to the optimality, a part of the terms in the right hand side of the above inequality,

$$\theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^*) \ge 0.$$

Using $Ax^* + By^* = b$ and $\lambda^k - \lambda^{k+1} = \beta(Ax^{k+1} + By^{k+1} - b)$ (see (4.6c)) to deal the last term in the right hand side of (4.15) , it follows that

$$
\begin{aligned}
&\beta \begin{pmatrix} x^{k+1} - x^* \\ y^{k+1} - y^* \end{pmatrix}^T \begin{pmatrix} A^T \\ B^T \end{pmatrix} B(y^k - y^{k+1}) \\
&= \beta[(Ax^{k+1} - Ax^*) + (By^{k+1} - By^*)]^T B(y^k - y^{k+1}) \\
&= (\lambda^k - \lambda^{k+1})^T B(y^k - y^{k+1}).
\end{aligned}
$$

The lemma is proved. □

**引理 7** *Let* $\{w^k\}$ *be the sequence generated by Linearized ADMM* (4.6) *for the problem* (1.1)*. Then, we have*

$$
(\lambda^k - \lambda^{k+1})^T B(y^k - y^{k+1}) \geq \frac{1}{2}\|y^k - y^{k+1}\|_{D_B}^2 - \frac{1}{2}\|y^{k-1} - y^k\|_{D_B}^2. \quad (4.16)
$$

**Proof**. First, (4.9) represents

$$
\begin{aligned}
y^{k+1} \in \mathcal{Y}, \;\; &\theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \\
&\{-B^T\lambda^{k+1} + D_B(y^{k+1} - y^k)\} \geq 0, \; \forall\, y \in \mathcal{Y}. \quad (4.17)
\end{aligned}
$$

Setting $k$ in (4.17) by $k-1$, we have

$$\begin{aligned} y^k \in \mathcal{Y}, \quad & \theta_2(y) - \theta_2(y^k) + (y-y^k)^T \\ & \{-B^T\lambda^k + D_B(y^k - y^{k-1})\} \geq 0, \quad \forall\, y \in \mathcal{Y}. \end{aligned} \tag{4.18}$$

Setting the $y$ in (4.17) and (4.18) by $y^k$ and $y^{k+1}$, respectively, and adding them, we get

$$(y^k - y^{k+1})^T \big\{ B^T(\lambda^k - \lambda^{k+1}) + D_B[(y^{k+1} - y^k) - (y^k - y^{k-1})] \big\} \geq 0.$$

From the above inequality we get

$$(y^k - y^{k+1})^T B^T(\lambda^k - \lambda^{k+1}) \geq (y^k - y^{k+1})^T D_B[(y^k - y^{k+1}) - (y^{k-1} - y^k)].$$

Using the Cauchy-Schwarz inequality for the right hand side term of the above inequality, we get (4.16) and the lemma is proved. $\Box$

By using Lemma 6 and Lemma 7, we can prove the following convergence theorem.

**定理 2** *Let* $\{w^k\}$ *be the sequence generated by Linearized ADMM* (4.6) *for the problem* (1.1)*. Then, we have*

$$\begin{aligned}&\big(\|v^{k+1}-v^*\|_G^2+\|y^k-y^{k+1}\|_{D_B}^2\big)\\&\quad\le\big(\|v^k-v^*\|_G^2+\|y^{k-1}-y^k\|_{D_B}^2\big)-\|v^k-v^{k+1}\|_G^2,\ \ \forall w^*\in\Omega^*, \qquad (4.19)\end{aligned}$$

*where* $G$ *is given by* (4.13)*.*

**Proof**. From Lemma 6 and Lemma 7, it follows that

$$(v^{k+1}-v^*)^TG(v^k-v^{k+1})\ge\frac{1}{2}\|y^k-y^{k+1}\|_{D_B}^2-\frac{1}{2}\|y^{k-1}-y^k\|_{D_B}^2,\quad\forall w^*\in\Omega^*.$$

Using the above inequality, for any $w^*\in\Omega^*$, we get

$$\begin{aligned}\|v^k-v^*\|_G^2&=\|(v^{k+1}-v^*)+(v^k-v^{k+1})\|_G^2\\&\ge\|v^{k+1}-v^*\|_G^2+\|v^k-v^{k+1}\|_G^2+2(v^{k+1}-v^*)^TG(v^k-v^{k+1})\\&\ge\|v^{k+1}-v^*\|_G^2+\|v^k-v^{k+1}\|_G^2\\&\qquad+\|y^k-y^{k+1}\|_{D_B}^2-\|y^{k-1}-y^k\|_{D_B}^2.\end{aligned}$$

The assertion of the Theorem 2 is proved. $\square$

## Optimal linearized ADMM – Main result in OO6228

In the subproblem of the Linearized ADMM, namely (4.6b), in order to ensure the convergence, it was required that

$$D_B = sI_{n_2} - \beta B^T B \quad \text{and} \quad s > \beta \|B^T B\|. \tag{4.20}$$

It is well known that the large parameter $s$ will lead a slow convergence.

Recent Advance in : Bingsheng He, Feng Ma, Xiaoming Yuan:
Optimally linearizing the alternating direction method of multipliers for convex programming, Comput. Optim. Appl. 75 (2020), 361-388.

We have proved: For the matrix $D_B$ in (4.6b) with form (4.20)

- if $s > \frac{3}{4}\beta\|B^T B\|$, the method is still convergent;
- if $s < \frac{3}{4}\beta\|B^T B\|$, there is divergent example.

# 5 Customized PPA for Variational Inequality

We study the algorithms using the guidance of variational inequality. The optimal condition of the linearly constrained convex optimization is resulted in a variational inequality:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{5.1}$$

## 5.1 Customized PPA for VI (5.1)

[Prediction Step.] With given $v^k$, find a vector $\tilde{w}^k \in \Omega$ which satisfying

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (v - \tilde{v}^k)^T H(v^k - \tilde{v}^k), \ \forall w \in \Omega, \tag{5.2a}$$

where the matrix $H$ is positive definite.

[Correction Step.] Determine a nonsingular matrix $M$ and a scalar $\alpha > 0$, let

$$v^{k+1} = v^k - \alpha(v^k - \tilde{v}^k), \quad \alpha \in (0, 2). \tag{5.2b}$$

$v$ is a part of the elements of the vector $w$, $\quad v = w$ is also possible.

## 5.2 Convergence proof

We prove the following main convergence property.

**定理 1** *Let* $\{v^k\}$ *be the sequence generated by* (5.2) *for the problem* (5.1) *and* $\tilde{w}^k$ *is obtained from* (5.2a)*. Then we have*

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \alpha(2-\alpha)\|v^k - \tilde{v}^k\|_H^2, \quad \forall v^* \in \mathcal{V}^*. \tag{5.3}$$

*where* $\mathcal{V}^* = \{v^* \mid v^* \text{ is a part of } w^*, w^* \in \Omega^*\}$*.*

**Proof**. Setting $w = w^*$ in (5.2a), we get

$$(\tilde{v}^k - v^*)^T H(v^k - \tilde{v}^k) \ge \theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(\tilde{w}^k), \ \forall w^* \in \Omega^*.$$

By using $(\tilde{w}^k - w^*)^T F(\tilde{w}^k) = (\tilde{w}^k - w^*)^T F(w^*)$ and the optimality of $w^*$, we have

$$(\tilde{v}^k - v^*)^T H(v^k - \tilde{v}^k) \ge 0, \quad \forall v^* \in \mathcal{V}^*.$$

It can be written as

$$\{(v^k - v^*) - (v^k - \tilde{v}^k)\}^T H(v^k - \tilde{v}^k) \ge 0, \quad \forall v^* \in \mathcal{V}^*,$$

and thus

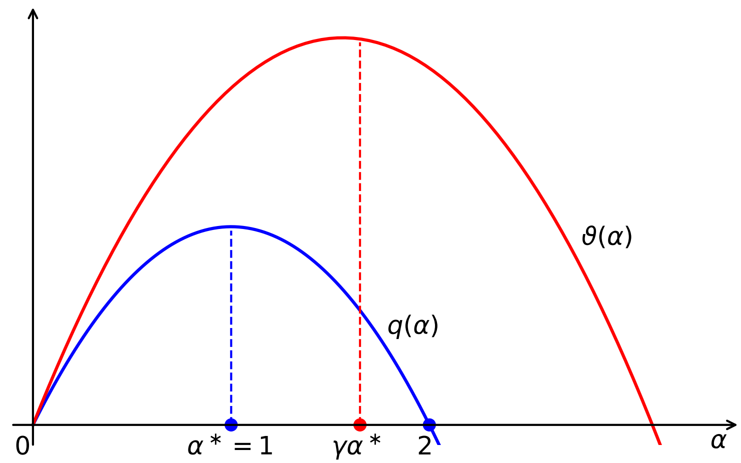
$$(v^k - v^*)^T H(v^k - \tilde{v}^k) \ge \|v^k - \tilde{v}^k\|_H^2, \quad \forall v^* \in \mathcal{V}^*. \tag{5.4}$$

Let

$$\vartheta(\alpha) = \|v^k - v^*\|_H^2 - \|v_\alpha^{k+1} - v^*\|_H^2.$$

It follows that

$$\begin{aligned}\vartheta(\alpha) &= \|v^k - v^*\|_H^2 - \|v_\alpha^{k+1} - v^*\|_H^2 \\ &= \|v^k - v^*\|_H^2 \\ &\quad -\|(v^k - v^*) - \alpha(v^k - \tilde{v}^k)\|_H^2 \\ &= 2\alpha(v^k - v^*)^T H(v^k - \tilde{v}^k) \\ &\quad -\alpha^2\|v^k - \tilde{v}^k\|_H^2. \qquad (5.5)\end{aligned}$$




取 $\gamma \in [1,2)$ 的示意图

Using (5.4), we get

$$\begin{aligned}\vartheta(\alpha) &\geq 2\alpha\|v^k - \tilde{v}^k\|_H^2 - \alpha^2\|v^k - \tilde{v}^k\|_H^2 \\ &:= q(\alpha) \qquad (5.6)\end{aligned}$$

The assertion (5.3) follows from (5.5) and (5.6) immediately. □

我们本想极大化 $\vartheta(\alpha)$, 虽然 $\vartheta(\alpha)$ 是 $\alpha$ 的二次函数, 但线性项系数 $2(v^k - v^*)^T H(v^k - \tilde{v}^k)$ 中含有未知的 $v^*$, 利用 (5.4), 得到 $\vartheta(\alpha)$ 的下界函数 $q(\alpha)$. 极大化 $q(\alpha)$, $\alpha_k^* \equiv 1$. 可以松弛延拓.

# 6 Applications for separable problems

## 6.1 ADMM in PPA-sense

根据 PPA 算法的要求　设计的右端矩阵为对称正定. 具体算法可参阅 [20]

In order to solve the separable convex optimization problem (1.1), we construct a method whose prediction-step is

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T H(v^k - \tilde{v}^k), \; \forall w \in \Omega, \tag{6.1a}$$

where

$$H = \begin{pmatrix} \beta B^T B + \delta I_{n_2} & -B^T \\ -B & \frac{1}{\beta} I_m \end{pmatrix}, \quad \text{(a small } \delta > 0\text{)}. \tag{6.1b}$$

Since $H$ is positive definite, we can use the update form of Algorithm I to produce the new iterate $v^{k+1} = (y^{k+1}, \lambda^{k+1})$. (In the algorithm [2], we took $\delta = 0$).

The concrete form of (6.1) is

$$\begin{cases} \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T \\ \qquad \{\underline{-A^T\tilde{\lambda}^k}\} \ge 0, \\ \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T \\ \qquad \{\underline{-B^T\tilde{\lambda}^k} + \boldsymbol{(\beta B^T B + \delta I_{n_2})}(\tilde{y}^k - y^k) \boldsymbol{- B^T}(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \\ (\underline{A\tilde{x}^k + B\tilde{y}^k - b}) \quad \boldsymbol{-B}(\tilde{y}^k - y^k) \quad + \quad \boldsymbol{(1/\beta)}\,(\tilde{\lambda}^k - \lambda^k) = 0. \end{cases}$$

The underline part is $F(\tilde{w}^k)$:

$$\boldsymbol{F(w)} = \begin{pmatrix} \boldsymbol{-A^T\lambda} \\ \boldsymbol{-B^T\lambda} \\ \boldsymbol{Ax + By - b} \end{pmatrix}$$

In fact, the prediction can be arranged by

$$\begin{cases} \tilde{x}^k \in \text{Argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{1}{2}\beta\|Ax + By^k - b\|^2 \,|\, x \in \mathcal{X}\}, & \text{(6.2a)} \\ \tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b), & \text{(6.2b)} \\ \tilde{y}^k \in \text{Argmin}\left\{ \begin{array}{c} \theta_2(y) - y^T B^T[2\tilde{\lambda}^k - \lambda^k] \\ + \frac{1}{2}\beta\|B(y - y^k)\|^2 + \frac{1}{2}\delta\|y - y^k\|^2 \end{array} \middle| \, y \in \mathcal{Y} \right\}. & \text{(6.2c)} \end{cases}$$

这个预测与经典的交替方向法 (3.3) 相当, 采用(5.2b) 校正, 会加快速度.

According to Lemma 1, the solution of (6.2a), $\tilde{x}^k$ satisfies

$$\tilde{x}^k \in \mathcal{X}, \quad \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{-A^T\lambda^k + \beta A^T(A\tilde{x}^k + By^k - b)\} \geq 0, \quad \forall x \in \mathcal{X}.$$

By using (6.2b), $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, the above variational inequality can be written as

$$\tilde{x}^k \in \mathcal{X}, \quad \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{\underline{-A^T\tilde{\lambda}^k}\} \geq 0, \quad \forall x \in \mathcal{X}.$$

The equation (6.2b) can written as

$$(\underline{A\tilde{x}^k + B\tilde{y}^k - b}) \boldsymbol{-B}(\tilde{y}^k - y^k) + \boldsymbol{(1/\beta)}(\tilde{\lambda}^k - \lambda^k) = 0.$$

The remainder part of the prediction (6.2c), namely,

$$\theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T \{\underline{-B^T\tilde{\lambda}^k} + \boldsymbol{(\beta B^T B + \delta I_{n_2})}(\tilde{y}^k - y^k) \boldsymbol{-B^T}(\tilde{\lambda}^k - \lambda^k)\} \geq 0$$

can be achieved by

$$\tilde{y}^k = \text{Argmin}\{\theta_2(y) - y^T B^T[2\tilde{\lambda}^k - \lambda^k] + \frac{1}{2}\beta\|B(y - y^k)\|^2 + \frac{1}{2}\delta\|y - y^k\|^2 \mid y \in \mathcal{Y}\}.$$

如果把 (6.2) 中取 $\delta = 0$, 并将其输出记为 $(x^{k+1}, \lambda^{k+1}, y^{k+1})$, 则迭代式为

$$\left\{\begin{array}{ll} x^{k+1} \in \text{Argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{\beta}{2}\|Ax + By^k - b\|^2 \mid x \in \mathcal{X}\}, & \text{(6.3a)} \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^k - b), & \text{(6.3b)} \\ y^{k+1} \in \text{Argmin}\big\{\theta_2(y) - y^T B^T [2\lambda^{k+1} - \lambda^k] + \frac{\beta}{2}\|B(y - y^k)\|^2 \bigm| y \in \mathcal{Y}\big\}. & \text{(6.3c)} \end{array}\right.$$

注意在 (6.3c) 中,

$$\begin{aligned} y^{k+1} &\in \text{Argmin}\big\{\theta_2(y) - y^T B^T [2\lambda^{k+1} - \lambda^k] + \frac{\beta}{2}\|B(y - y^k)\|^2 \bigm| y \in \mathcal{Y}\big\} \\ &= \text{Argmin}\big\{\theta_2(y) - y^T B^T \lambda^{k+1} - y^T B^T (\lambda^{k+1} - \lambda^k) + \frac{\beta}{2}\|B(y - y^k)\|^2 \bigm| y \in \mathcal{Y}\big\} \\ &= \text{Argmin}\big\{\theta_2(y) - y^T B^T \lambda^{k+1} + \frac{\beta}{2}\|B(y - y^k) - \frac{1}{\beta}(\lambda^{k+1} - \lambda^k)\|^2 \bigm| y \in \mathcal{Y}\big\} \\ &= \text{Argmin}\big\{\theta_2(y) - y^T B^T \lambda^{k+1} + \frac{\beta}{2}\|B(y - y^k) + \frac{1}{\beta}(\lambda^k - \lambda^{k+1})\|^2 \bigm| y \in \mathcal{Y}\big\} \\ &= \text{Argmin}\big\{\theta_2(y) - y^T B^T \lambda^{k+1} + \frac{\beta}{2}\|B(y - y^k) + (Ax^{k+1} + By^k - b)\|^2 \bigm| y \in \mathcal{Y}\big\} \\ &= \text{Argmin}\big\{\theta_2(y) - y^T B^T \lambda^{k+1} + \frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \bigm| y \in \mathcal{Y}\big\}. \end{aligned}$$

所以, (6.3) 就是

$$
\left\{
\begin{array}{ll}
x^{k+1} \in \text{Argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{\beta}{2}\|Ax + By^k - b\|^2 \mid x \in \mathcal{X}\}, & \text{(6.4a)} \\
\lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^k - b), & \text{(6.4b)} \\
y^{k+1} \in \text{Argmin}\{\theta_2(y) - y^T B^T \lambda^{k+1} + \frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \mid y \in \mathcal{Y}\}. & \text{(6.4c)}
\end{array}
\right.
$$

请注意, 经典的 ADMM 是

$$
\begin{array}{lcl}
x^{k+1} & \in & \text{Argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{\beta}{2}\|Ax + By^k - b\|^2 \mid x \in \mathcal{X}\}, \\
y^{k+1} & \in & \text{Argmin}\{\theta_2(y) - y^T B^T \lambda^k + \frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \mid y \in \mathcal{Y}\}, \\
\lambda^{k+1} & = & \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b).
\end{array}
$$

所以, (6.3), 就是交换了 $y, \lambda$ 顺序的交替方向法. 由于可以采用

$$v^{k+1} := v^k - \alpha(v^k - v^{k+1}), \quad \alpha \in (0, 2).$$

通常取 $\alpha = 1.5$, 收敛更快.

## 6.2 Linearized ADMM-Like Method

当子问题 (6.2c) 求解有困难时, 用 $\frac{s}{2}\|y-y^k\|^2$ 代替 $\frac{1}{2}\|y-y^k\|^2_{(\beta B^TB+\delta I_{n_2})}$

By using the linearized version of (6.2c), the prediction step becomes

$$\theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^TF(\tilde{w}^k)\ge(v-\tilde{v}^k)^TH(v^k-\tilde{v}^k),\ \forall w\in\Omega, \quad (6.5)$$

where

$$H=\begin{bmatrix} sI & -B^T\\ -B & \frac{1}{\beta}I_m\end{bmatrix}, \text{ 代替 (6.1) 中的 } \begin{bmatrix}\beta B^TB+\delta I_{n_2} & -B^T\\ -B & \frac{1}{\beta}I_m\end{bmatrix}. \quad (6.6)$$

The concrete formula of (6.5) is

The underline part is $F(\tilde{w}^k)$:

$$\boldsymbol{F(w)}=\begin{pmatrix}-\boldsymbol{A^T\lambda}\\ -\boldsymbol{B^T\lambda}\\ \boldsymbol{Ax+By-b}\end{pmatrix}$$

$$\left\{\begin{array}{l}\theta_1(x)-\theta_1(\tilde{x}^k)+(x-\tilde{x}^k)^T\\ \qquad\qquad \{\underline{-A^T\tilde{\lambda}^k}\}\ge 0,\\ \theta_2(y)-\theta_2(\tilde{y}^k)+(y-\tilde{y}^k)^T\\ \qquad\qquad \{\underline{-B^T\tilde{\lambda}^k}+\ \boldsymbol{s}(\tilde{y}^k-y^k)\boldsymbol{-B^T}(\tilde{\lambda}^k-\lambda^k)\}\ge 0,\\ (\underline{A\tilde{x}^k+B\tilde{y}^k-b})\boldsymbol{-B}(\tilde{y}^k-y^k)+\boldsymbol{(1/\beta)}(\tilde{\lambda}^k-\lambda^k)=0.\end{array}\right. \quad (6.7)$$

**How to implement the prediction?** To get $\tilde{w}^k$ which satisfies (6.7),

we need only use the following procedure:

$$\begin{cases} \tilde{x}^k \in \text{Argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{1}{2}\beta\|Ax + By^k - b\|^2 \,|\, x \in \mathcal{X}\}, \\ \tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b), \\ \tilde{y}^k = \text{Argmin}\{\theta_2(y) - y^T B^T [\mathbf{2\tilde{\lambda}^k - \lambda^k}] + \dfrac{s}{2}\|y - y^k\|^2 \,|\, y \in \mathcal{Y}\}. \end{cases} \tag{6.8}$$

用 $\frac{s}{2}\|y - y^k\|^2$ 代替 $\frac{1}{2}(\beta\|B(y - y^k)\|^2 + \delta\|y - y^k\|^2)$, 为保证收敛,

需要 $s > \beta\|B^T B\|$. 对给定的 $\beta > 0$, 太大的 $s$ 会影响收敛速度.

只有当由二次项 $\frac{1}{2}\beta\|B(y - y^k)\|^2$ 引发求解困难, 才用线性化方法.

Then, we use the form

$$v^{k+1} = v^k - \alpha(v^k - \tilde{v}^k), \quad \alpha \in (0, 2)$$

to update the new iterate $v^{k+1}$.

# 7 Solving the primal subproblem in parallel

根据 PPA 算法的要求　设计的右端矩阵为对称正定.

Primal-Dual Order

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (w - \tilde{w}^k)^T H(w^k - \tilde{w}^k), \ \forall w \in \Omega, \tag{7.1a}$$

where

$$H = \begin{pmatrix} \beta A^T A + \delta I_{n_1} & 0 & A^T \\ 0 & \beta B^T B + \delta I_{n_2} & B^T \\ A & B & \frac{2}{\beta} I_m \end{pmatrix}. \tag{7.1b}$$

The both matrices

$$\begin{pmatrix} \beta A^T A + \delta I_{n_1} & A^T \\ A & \frac{1}{\beta} I_m \end{pmatrix} \succ 0, \qquad \begin{pmatrix} \beta B^T B + \delta I_{n_2} & B^T \\ B & \frac{1}{\beta} I_m \end{pmatrix} \succ 0.$$

The concrete form of (7.1) is

$$\left\{\begin{array}{l} \theta_1(x)-\theta_1(\tilde{x}^k)+(x-\tilde{x}^k)^T \\ \qquad\qquad \{\underline{-A^T\tilde{\lambda}^k}+\boldsymbol{(\beta A^T A+\delta I_{n_1})}(\tilde{x}^k-x^k)\boldsymbol{+A^T}(\tilde{\lambda}^k-\lambda^k)\}\ge 0, \\ \theta_2(y)-\theta_2(\tilde{y}^k)+(y-\tilde{y}^k)^T \\ \qquad\qquad \{\underline{-B^T\tilde{\lambda}^k}+\boldsymbol{(\beta B^T B+\delta I_{n_2})}(\tilde{y}^k-y^k)\boldsymbol{+B^T}(\tilde{\lambda}^k-\lambda^k)\}\ge 0, \\ (\underline{A\tilde{x}^k+B\tilde{y}^k-b})\boldsymbol{+A}(\tilde{x}^k-x^k)\boldsymbol{+B}(\tilde{y}^k-y^k)+\boldsymbol{(2/\beta)}\,(\tilde{\lambda}^k-\lambda^k)=0. \end{array}\right.$$

整理一下得到

$$\left\{\begin{array}{l} \theta_1(x)-\theta_1(\tilde{x}^k)+(x-\tilde{x}^k)^T\{-A^T\lambda^k+(\beta A^T A+\delta I_{n_1})(\tilde{x}^k-x^k)\}\ge 0, \\ \theta_2(y)-\theta_2(\tilde{y}^k)+(y-\tilde{y}^k)^T\{-B^T\lambda^k+(\beta B^T B+\delta I_{n_2})(\tilde{y}^k-y^k)\}\ge 0, \\ [2(A\tilde{x}^k+B\tilde{y}^k-b)-(Ax^k+By^k-b)]+(2/\beta)(\tilde{\lambda}^k-\lambda^k)=0. \end{array}\right.$$

In fact, the prediction can be arranged by

$$\begin{cases} \tilde{x}^k = \arg\min\left\{ \begin{array}{c} \theta_1(x) - x^T A^T \lambda^k \\ +\frac{1}{2}\beta\|A(x-x^k)\|^2 + \frac{1}{2}\delta\|x-x^k\|^2 \end{array} \middle| x \in \mathcal{X} \right\} & (7.2a) \\ \tilde{y}^k = \arg\min\left\{ \begin{array}{c} \theta_2(y) - y^T B^T \lambda^k \\ +\frac{1}{2}\beta\|B(y-y^k)\|^2 + \frac{1}{2}\delta\|y-y^k\|^2 \end{array} \middle| y \in \mathcal{Y} \right\} & (7.2b) \\ \tilde{\lambda}^k = \lambda^k - \frac{1}{2}\beta\big[2(A\tilde{x}^k + B\tilde{y}^k - b) - (Ax^k + By^k - b)\big] & (7.2c) \end{cases}$$

$$\begin{cases} \tilde{x}^k = \arg\min\{\theta_1(x) - x^T A^T \lambda^k + \frac{1}{2}(x-x^k)^T(\beta A^T A + \delta I_{n_1})(x-x^k) | x \in \mathcal{X}\} \\ \tilde{y}^k = \arg\min\{\theta_2(y) - y^T B^T \lambda^k + \frac{1}{2}(y-y^k)^T(\beta B^T B + \delta I_{n_2})(y-y^k) | y \in \mathcal{Y}\} \\ \tilde{\lambda}^k = \lambda^k - \frac{1}{2}\beta\big[2(A\tilde{x}^k + B\tilde{y}^k - b) - (Ax^k + By^k - b)\big] \end{cases}$$

$$w^{k+1} = w^k - \alpha(w^k - \tilde{w}^k), \qquad \alpha \in (0, 2).$$

## Dual-Primal Order

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (w - \tilde{w}^k)^T H(w^k - \tilde{w}^k), \ \forall w \in \Omega, \tag{7.3a}$$

where

$$H = \begin{pmatrix} \beta A^T A + \delta I_{n_1} & 0 & -A^T \\ 0 & \beta B^T B + \delta I_{n_2} & -B^T \\ -A & -B & \frac{2}{\beta} I_m \end{pmatrix}. \tag{7.3b}$$

The both matrices

$$H = \begin{pmatrix} \beta A^T A + \delta I_{n_1} & -A^T \\ -A & \frac{1}{\beta} I_m \end{pmatrix} \succ 0, \quad \begin{pmatrix} \beta B^T B + \delta I_{n_2} & -B^T \\ -B & \frac{1}{\beta} I_m \end{pmatrix} \succ 0.$$

The concrete form of (7.3) is

$$\left\{\begin{array}{l}\theta_1(x)-\theta_1(\tilde{x}^k)+(x-\tilde{x}^k)^T \\ \qquad\qquad \{\underline{-A^T\tilde{\lambda}^k}+\boldsymbol{(\beta A^T A+\delta I_{n_1})}(\tilde{x}^k-x^k)\boldsymbol{-A^T}(\tilde{\lambda}^k-\lambda^k)\}\ge 0, \\ \theta_2(y)-\theta_2(\tilde{y}^k)+(y-\tilde{y}^k)^T \\ \qquad\qquad \{\underline{-B^T\tilde{\lambda}^k}+\boldsymbol{(\beta B^T B+\delta I_{n_2})}(\tilde{y}^k-y^k)\boldsymbol{-B^T}(\tilde{\lambda}^k-\lambda^k)\}\ge 0, \\ (\underline{A\tilde{x}^k+B\tilde{y}^k-b})\boldsymbol{-A}(\tilde{x}^k-x^k)\boldsymbol{-B}(\tilde{y}^k-y^k)+\boldsymbol{(2/\beta)}\,(\tilde{\lambda}^k-\lambda^k)=0.\end{array}\right.$$

经整理归并一下得到

$$\left\{\begin{array}{l}\theta_1(x)-\theta_1(\tilde{x}^k)+(x-\tilde{x}^k)^T\{-A^T(2\tilde{\lambda}^k-\lambda^k) \\ \qquad\qquad\qquad\qquad +(\beta A^T A+\delta I_{n_1})(\tilde{x}^k-x^k)\}\ge 0, \\ \theta_2(y)-\theta_2(\tilde{y}^k)+(y-\tilde{y}^k)^T\{-B^T(2\tilde{\lambda}^k-\lambda^k) \\ \qquad\qquad\qquad\qquad +(\beta B^T B+\delta I_{n_2})(\tilde{y}^k-y^k)\}\ge 0, \\ (Ax^k+By^k-b)+(2/\beta)(\tilde{\lambda}^k-\lambda^k)=0.\end{array}\right.$$

In fact, the prediction can be arranged by

$$
\begin{cases}
\tilde{\lambda}^k = \lambda^k - \frac{1}{2}\beta(Ax^k + By^k - b), & \text{(7.4a)} \\[2ex]
\tilde{x}^k \in \arg\min\left\{ \begin{array}{c} \theta_1(x) - x^T A^T [2\tilde{\lambda}^k - \lambda^k] \\ +\frac{1}{2}\beta\|A(x - x^k)\|^2 + \frac{1}{2}\delta\|x - x^k\|^2 \end{array} \middle| \, x \in \mathcal{X} \right\} & \text{(7.4b)} \\[2ex]
\tilde{y}^k \in \arg\min\left\{ \begin{array}{c} \theta_2(y) - y^T B^T [2\tilde{\lambda}^k - \lambda^k] \\ +\frac{1}{2}\beta\|B(y - y^k)\|^2 + \frac{1}{2}\delta\|y - y^k\|^2 \end{array} \middle| \, y \in \mathcal{Y} \right\}. & \text{(7.4c)}
\end{cases}
$$

$$
w^{k+1} = w^k - \alpha(w^k - \tilde{w}^k), \qquad \alpha \in (0, 2).
$$

我们关于ADMM的研究, 始于1997年, 第一篇ADMM方面的论文发表于1998年. 这一讲中§4-§6介绍的ADMM类方法, 可以从[20]中找到.

利用变分不等式(VI)和邻近点算法(PPA), 更自由地设计ADMM类分裂收缩算法

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