变分不等式框架下结构型凸优化的分裂收缩算法

IV. 线性约束凸优化问题分裂收缩算法的统一框架

中学的数理基础　必要的社会实践
普通的大学数学　一般的优化原理

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1 凸优化分裂收缩算法的统一框架

我们总是用变分不等式 (VI) 指导算法设计, 把线性约束的凸优化问题归结为下面的变分不等式:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{1.1}$$

Algorithms in a unified framework

A unified Algorithmic Framework for (1.1)

统一框架由预测-校正两部分组成

[Prediction Step.] 从给定的 v^k 出发, 求得预测点 $\tilde{w}^k \in \Omega$ 使其满足

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \ \forall w \in \Omega, \tag{1.2a}$$

其中 Q 不一定对称, 但是 $Q^T + Q$ 正定.

[Correction Step.] 给一个合适的非奇异矩阵 M, 由下式确定新的迭代点

$$v^{k+1} = v^k - M(v^k - \tilde{v}^k). \tag{1.2b}$$

Q 和 M 分别叫做预测矩阵和校正矩阵

Convergence Conditions

For the matrices Q and M, there is a positive definite matrix H such that

$$HM = Q. \tag{1.3a}$$

In addition,

$$G = Q^T + Q - M^T HM \succ 0. \tag{1.3b}$$

其实, 只要预测 (1.2a) 中的预测矩阵 Q 满足

$$Q^T + Q \succ 0,$$

我们总可以取

$$0 \prec G \prec Q^T + Q.$$

然后记

$$D = (Q^T + Q) - G,$$

则 $D \succ 0$. 令

$$M^T HM = D.$$

由矩阵方程组解得

$$\left\{\begin{array}{r} HM = Q, \\ M^T HM = D. \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{r} HM = Q, \\ Q^T M = D. \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} H = QD^{-1}Q^T, \\ M = Q^{-T}D. \end{array}\right.$$

就得到满足收敛条件的校正矩阵 M.

实际计算中, 我们只要校正矩阵 M.

H 和 G 只是用来验证收敛条件的.

换句话说, 只要

$$Q^T + Q \succ 0.$$

我们就可以选两个正定矩阵 $D \succ 0$ 和 $G \succ 0$, 使得

$$D + G = Q^T + Q.$$

将 (1.2b) 中的校正矩阵 M 取成

$$M = Q^{-T} D$$

条件 (1.3) 自然满足.

2 预测-校正方法的例子

We consider the $\min - \max$ problem

$$\min_x \max_y \{\Phi(x,y) = \theta_1(x) - y^T Ax - \theta_2(y) \,|\, x \in \mathcal{X}, y \in \mathcal{Y}\}. \tag{2.4}$$

Let (x^*, y^*) be the solution of (2.4), then we have 根据鞍点的定义

$$(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}, \quad \Phi(x^*, y) \le \Phi(x^*, y^*) \le \Phi(x, y^*), \quad \forall\, (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

上面的两个不等式可以写成等价的

$$\left\{ \begin{array}{ll} x^* \in \mathcal{X}, \quad \Phi(x, y^*) - \Phi(x^*, y^*) \ge 0, \quad \forall\, x \in \mathcal{X}, & \text{(2.5a)} \\ y^* \in \mathcal{Y}, \quad \Phi(x^*, y^*) - \Phi(x^*, y) \ge 0, \quad \forall\, y \in \mathcal{Y}. & \text{(2.5b)} \end{array} \right.$$

Using the notation of $\Phi(x,y)$, it can be written as 只要把 $\Phi(x,y)$ 的形式填进去

$$\left\{ \begin{array}{llll} x^* \in \mathcal{X}, & \theta_1(x) - \theta_1(x^*) + (x - x^*)^T(-A^T y^*) \ge 0, & \forall\, x \in \mathcal{X}, & (*) \\ y^* \in \mathcal{Y}, & \theta_2(y) - \theta_2(y^*) \; + \; (y - y^*)^T(\; Ax^*) \;\; \ge 0, & \forall\, y \in \mathcal{Y}. & (\diamond) \end{array} \right.$$

Furthermore, it can be written as a variational inequality in the compact form:

$$u \in \Omega, \quad \theta(u) - \theta(u^*) + (u - u^*)^T F(u^*) \geq 0, \ \forall\, u \in \Omega, \tag{2.6}$$

where 对上式中任意的 $u \in \Omega$ 分别取 $u = (x, y^*)$ 和 $u = (x^*, y)$, 就得到 $(*)$ 和 $(\diamond)$.

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y), \quad F(u) = \begin{pmatrix} -A^T y \\ Ax \end{pmatrix}, \quad \Omega = \mathcal{X} \times \mathcal{Y}.$$

The output of Original PDHG algorithm [16] as predictor

For given (x^k, y^k), PDHG [16] produces a pair of $(\tilde{x}^k, \tilde{y}^k)$. First,

$$\tilde{x}^k = \operatorname{argmin}\{\Phi(x, y^k) + \frac{r}{2}\|x - x^k\|^2 \,|\, x \in \mathcal{X}\}, \tag{2.7a}$$

and then we obtain $\tilde{y}^k$ via

$$\tilde{y}^k = \operatorname{argmax}\{\Phi(\tilde{x}^k, y) - \frac{s}{2}\|y - y^k\|^2 \,|\, y \in \mathcal{Y}\}. \tag{2.7b}$$

Ignoring the constant term in the objective function, the subproblems (2.7) are reduced to

$$\left\{ \begin{array}{ll} \tilde{x}^k = \operatorname{argmin}\{\theta_1(x) - x^T A^T y^k + \frac{r}{2}\|x - x^k\|^2 \,|\, x \in \mathcal{X}\}, & \text{(2.8a)} \\ \tilde{y}^k = \operatorname{argmin}\{\theta_2(y) + y^T A \tilde{x}^k + \frac{s}{2}\|y - y^k\|^2 \,|\, y \in \mathcal{Y}\}. & \text{(2.8b)} \end{array} \right.$$

According to the basic lemma, the optimality condition of (2.8a) is $\tilde{x}^k \in \mathcal{X}$ and

$$\theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{-A^T y^k + r(\tilde{x}^k - x^k)\} \geq 0, \;\forall\, x \in \mathcal{X}. \quad (2.9)$$

Similarly, from (2.8b) we get $\tilde{y}^k \in \mathcal{Y}$ and

$$\theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T\{A\tilde{x}^k + s(\tilde{y}^k - y^k)\} \geq 0, \;\forall\, y \in \mathcal{Y}. \quad (2.10)$$

Combining (2.9) and (2.10), we have

$$\tilde{u}^k \in \Omega, \quad \theta(u) - \theta(\tilde{u}^k) + \begin{pmatrix} x - \tilde{x}^k \\ y - \tilde{y}^k \end{pmatrix}^T \left\{ \begin{pmatrix} -A^T \tilde{y}^k \\ A\tilde{x}^k \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} r(\tilde{x}^k - x^k) + A^T(\tilde{y}^k - y^k) \\ s(\tilde{y}^k - y^k) \end{pmatrix} \right\} \geq 0, \quad \forall (x, y) \in \Omega.$$

The compact form is $\tilde{u}^k \in \Omega$,

$$\theta(u) - \theta(\tilde{u}^k) + (u - \tilde{u}^k)^T \{F(\tilde{u}^k) + Q(\tilde{u}^k - u^k)\} \geq 0, \ \forall u \in \Omega, \tag{2.11a}$$

where

$$Q = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix}. \tag{2.11b}$$

对于这样的预测, 我们考虑比较简单的校正

$$u^{k+1} = u^k - M(u^k - \tilde{u}^k) \tag{2.12}$$

校正. 其中 M 为单位上三角矩阵或单位下三角矩阵. 收敛性条件 (1.3)

- $H \succ 0$ and $HM = Q$.
- $G = Q^T + Q - M^T HM \succ 0$.

可以改写成等价的

(i) $H \succ 0$ and $H = QM^{-1}$.

(ii) $G = Q^T + Q - Q^T M \succ 0$.

一. 校正矩阵 M 为单位下三角矩阵

其中的 K 是待定的.

$$M = \begin{pmatrix} I_n & 0 \\ K & I_m \end{pmatrix} \quad \text{则} \quad M^{-1} = \begin{pmatrix} I_n & 0 \\ -K & I_m \end{pmatrix}.$$

对条件 (i), 我们在统一框架下指导下求出这个 K 的具体形式. 由于 $H = QM^{-1}$ 正定, 首先必须是对称的. 由

$$H = QM^{-1} = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -K & I_m \end{pmatrix} = \begin{pmatrix} rI_n - A^T K & A^T \\ -sK & sI_m \end{pmatrix}$$

必须对称, 推得

$$-sK = A, \qquad \Rightarrow \qquad K = -\frac{1}{s}A.$$

因此,

$$M = \begin{pmatrix} I_n & 0 \\ -\frac{1}{s}A & I_m \end{pmatrix}, \qquad H = \begin{pmatrix} rI_n + \frac{1}{s}A^T A & A^T \\ A & sI_m \end{pmatrix}.$$

对任意的 $r, s > 0$, 矩阵 H 是正定的.

对条件 (ii),

$$\begin{aligned} G &= Q^T + Q - M^T H M = Q^T + Q - Q^T M \\ &= \begin{pmatrix} 2rI_n & A^T \\ A & 2sI_m \end{pmatrix} - \begin{pmatrix} rI_n & 0 \\ A & sI_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -\frac{1}{s}A & I_m \end{pmatrix} \\ &= \begin{pmatrix} 2rI_n & A^T \\ A & 2sI_m \end{pmatrix} - \begin{pmatrix} rI_n & 0 \\ 0 & sI_m \end{pmatrix} = \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix}. \end{aligned}$$

要矩阵 G 正定, 必须有 $rs > \|A^T A\|$.

采用 PDHG 预测, 单位下三角矩阵校正, 需要 $rs > \|A^T A\|$.

二. 校正矩阵 $\boldsymbol{M}$ 为单位上三角矩阵

同样, 其中的 K 是待定的.

$$M = \begin{pmatrix} I_n & K \\ 0 & I_m \end{pmatrix} \quad \text{则} \quad M^{-1} = \begin{pmatrix} I_n & -K \\ 0 & I_m \end{pmatrix}.$$

对条件 (i), 我们在统一框架下指导下求出这个 K 的具体形式. 由于 $H = QM^{-1}$ 正定,

首先必须是对称的. 由

$$H = QM^{-1} = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix} \begin{pmatrix} I_n & -K \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} rI_n & -rK + A^T \\ 0 & sI_m \end{pmatrix}$$

必须对称, 推得

$$rK = A^T, \qquad \Rightarrow \qquad K = \frac{1}{r} A^T.$$

因此,

$$M = \begin{pmatrix} I_n & \frac{1}{r} A^T \\ 0 & I_m \end{pmatrix}, \qquad H = \begin{pmatrix} rI_n & 0 \\ 0 & sI_m \end{pmatrix}.$$

对任意的 $r, s > 0$, 矩阵 H 是正定的.

而对条件 (ii),

$$
\begin{aligned}
G &= Q^T + Q - M^T H M = Q^T + Q - Q^T M \\
&= \begin{pmatrix} 2rI_n & A^T \\ A & 2sI_m \end{pmatrix} - \begin{pmatrix} rI_n & 0 \\ A & sI_m \end{pmatrix} \begin{pmatrix} I_n & \frac{1}{r}A^T \\ 0 & I_m \end{pmatrix} \\
&= \begin{pmatrix} 2rI_n & A^T \\ A & 2sI_m \end{pmatrix} - \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix} \\
&= \begin{pmatrix} rI_n & 0 \\ 0 & sI_m - \frac{1}{r}AA^T \end{pmatrix}.
\end{aligned}
$$

要矩阵 G 正定, 必须有 $rs > \|A^T A\|$.

采用 PDHG 预测, 单位上三角矩阵校正, 需要 $rs > \|A^T A\|$.

虽然把不能保证收敛的 PDHG 方法改造成了收敛的方法, 但是, rs 的值没有降下来.

我们的目标, 是把预测 (2.8) 中的参数 rs 想办法降下来.

对于 (2.11) 中的 Q, 我们有

$$
Q^T + Q = \begin{pmatrix} 2rI & A^T \\ A & 2sI \end{pmatrix}
$$

只要 $rs > \frac{1}{4}\|A^TA\|$, 矩阵 $Q^T + Q$ 都是正定的.

当 $(Q^T + Q)$ 正定时, 我们取

$$D = \frac{1}{2}(Q^T + Q), \qquad \text{并令} \qquad M^T H M = D. \tag{2.13}$$

这样就能保证

$$G = Q^T + Q - M^T H M = \frac{1}{2}(Q^T + Q) \succ 0.$$

• $H \succ 0$ and $HM = Q$. • $G = Q^T + Q - M^T H M \succ 0$.	可以改写成	(i) $HM = Q$. (ii) $M^T H M = D$.

$$\left\{\begin{array}{r} HM = Q, \\ M^T H M = D. \end{array}\right. \Leftrightarrow \left\{\begin{array}{r} HM = Q, \\ Q^T M = D. \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} H = Q D^{-1} Q^T, \\ M = Q^{-T} D. \end{array}\right. \tag{2.14}$$

换句话说, 当 $(Q^T + Q) \succ 0$, 取

$$D = \begin{pmatrix} rI & \frac{1}{2}A^T \\ \frac{1}{2}A & sI \end{pmatrix}, \quad M = Q^{-T} D$$

所有收敛性条件都满足. 而

$$Q^{-T} = \begin{pmatrix} rI & 0 \\ A & sI \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{r}I & 0 \\ -\frac{1}{rs}A & \frac{1}{s}I \end{pmatrix}$$

$$\begin{aligned} M &= Q^{-T}D = \begin{pmatrix} \frac{1}{r}I & 0 \\ -\frac{1}{rs}A & \frac{1}{s}I \end{pmatrix} \begin{pmatrix} rI & \frac{1}{2}A^T \\ \frac{1}{2}A & sI \end{pmatrix} \\ &= \begin{pmatrix} I & \frac{1}{2r}A^T \\ -\frac{1}{2s}A & I - \frac{1}{2rs}AA^T \end{pmatrix} \end{aligned} \tag{2.15}$$

利用上面的校正矩阵 M

$$\begin{cases} x^{k+1} = \tilde{x}^k - \frac{1}{2r}A^T(y^k - \tilde{y}^k) \\ y^{k+1} = \tilde{y}^k + \frac{1}{2s}A[(x^k - \tilde{x}^k) + \frac{1}{r}A^T(y^k - \tilde{y}^k)]. \end{cases}$$

这是马峰他们 [14] 根据统一框架提出的方法. 计算效果有很大进步.

$\boxed{\text{把 } rs \text{ 的积降了 } \frac{3}{4}\text{, 有了很大进步.}}$

3 Convergence proof in the unified framework

In this section, assuming the conditions (1.3) in the unified framework are satisfied, we prove some convergence properties.

定理 1 *Let* $\{v^k\}$ *be the sequence generated by a method for the problem* (1.1) *and* $\tilde{w}^k$ *is obtained in the* k*-th iteration. If* v^k, v^{k+1} *and* $\tilde{w}^k$ *satisfy the conditions in the unified framework, then we have*

$$\begin{aligned} &\theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^TF(\tilde{w}^k) \\ \geq\quad & \frac{1}{2}\big(\|v-v^{k+1}\|_H^2-\|v-v^k\|_H^2\big)+\frac{1}{2}\|v^k-\tilde{v}^k\|_G^2, \quad \forall w\in\Omega. \end{aligned} \tag{3.1}$$

Proof. Using $Q=HM$ (see (1.3a)) and the relation (1.2b), the right hand side of (1.3a) can be written as $(v-\tilde{v}^k)^TH(v^k-v^{k+1})$ and hence

$$\theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^TF(\tilde{w}^k)\geq(v-\tilde{v}^k)^TH(v^k-v^{k+1}), \ \forall w\in\Omega. \tag{3.2}$$

Applying the identity

$$\boxed{Q(v^k-\tilde{v}^k)=HM(v^k-\tilde{v}^k)=H(v^k-v^{k+1}).}$$

$$(a-b)^TH(c-d)=\frac{1}{2}\{\|a-d\|_H^2-\|a-c\|_H^2\}+\frac{1}{2}\{\|c-b\|_H^2-\|d-b\|_H^2\},$$

to the right hand side of (3.2) with

$$a = v, \quad b = \tilde{v}^k, \quad c = v^k, \quad \text{and} \quad d = v^{k+1},$$

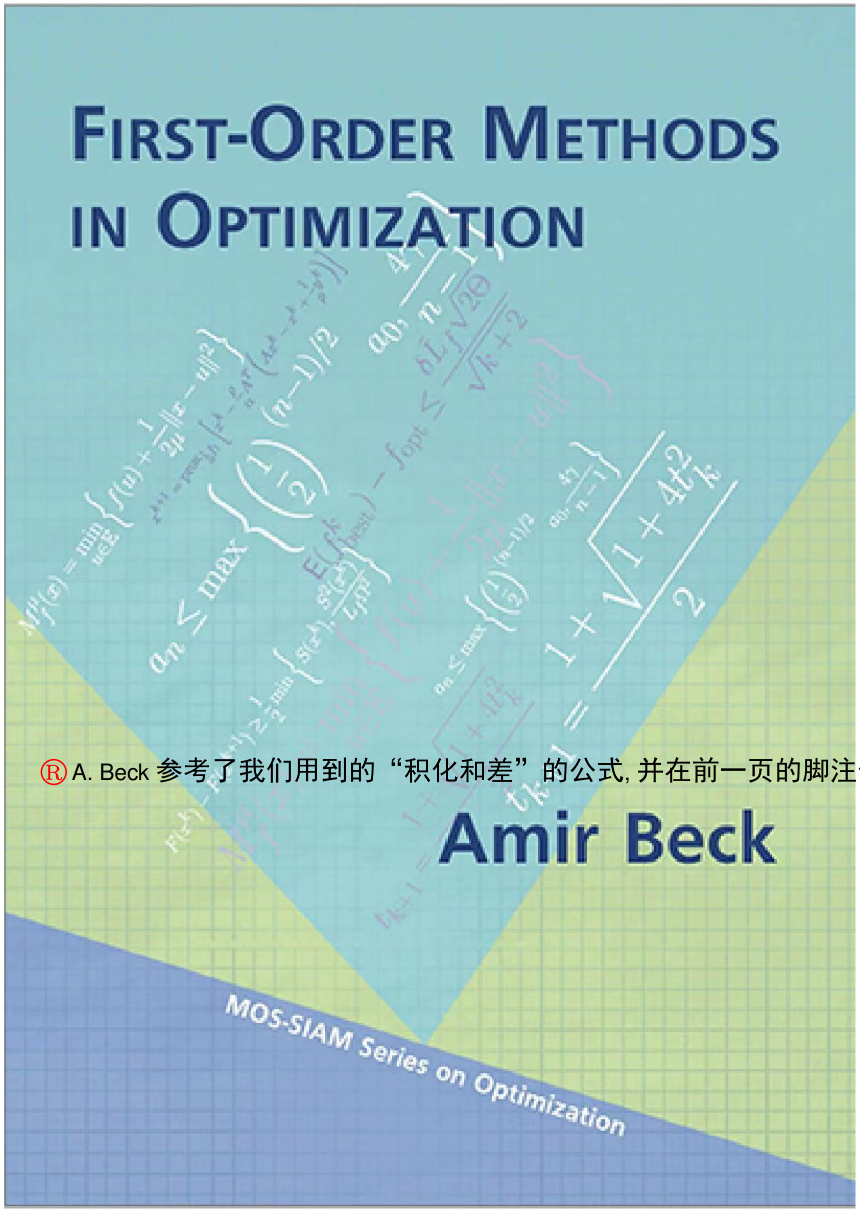
we thus obtain

$$\begin{aligned} & 2(v - \tilde{v}^k)^T H(v^k - v^{k+1}) \\ & \quad = \big(\|v - v^{k+1}\|_H^2 - \|v - v^k\|_H^2\big) + (\|v^k - \tilde{v}^k\|_H^2 - \|v^{k+1} - \tilde{v}^k\|_H^2). \end{aligned} \quad (3.3)$$

For the last term of (3.3), using $HM = Q$ and $2v^T Qv = v^T(Q^T + Q)v$, we have

$$\begin{aligned} & \|v^k - \tilde{v}^k\|_H^2 - \|v^{k+1} - \tilde{v}^k\|_H^2 \\ & \quad = \quad \|v^k - \tilde{v}^k\|_H^2 - \|(v^k - \tilde{v}^k) - (v^k - v^{k+1})\|_H^2 \\ & \quad \overset{(1.3a)}{=} \quad \|v^k - \tilde{v}^k\|_H^2 - \|(v^k - \tilde{v}^k) - M(v^k - \tilde{v}^k)\|_H^2 \\ & \quad = \quad 2(v^k - \tilde{v}^k)^T HM(v^k - \tilde{v}^k) - (v^k - \tilde{v}^k)^T M^T HM(v^k - \tilde{v}^k) \\ & \quad = \quad (v^k - \tilde{v}^k)^T (Q^T + Q - M^T HM)(v^k - \tilde{v}^k) \\ & \quad \overset{(1.3b)}{=} \quad \|v^k - \tilde{v}^k\|_G^2. \end{aligned} \quad (3.4)$$

Substituting (3.3), (3.4) in (3.2), the assertion of this theorem is proved. $\square$



Ⓡ A. Beck 参考了我们用到的“积化和差”的公式, 并在前一页的脚注做了说明

We will use the following notation:

$$\boxed{\begin{aligned}\tilde{\mathbf{x}}^k &= \mathbf{x}^{k+1},\\ \tilde{\mathbf{z}}^k &= \mathbf{z}^{k+1},\\ \tilde{\mathbf{y}}^k &= \mathbf{y}^k + \rho(\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{z}^k - \mathbf{c}).\end{aligned}}$$

Using (15.15), (15.16), the subgradient inequality, and the above notation, we obtain that for any $\mathbf{x} \in \mathrm{dom}(h_1)$ and $\mathbf{z} \in \mathrm{dom}(h_2)$,

$$h_1(\mathbf{x}) - h_1(\tilde{\mathbf{x}}^k) + \left\langle \rho\mathbf{A}^T\left(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c} + \frac{1}{\rho}\mathbf{y}^k\right) + \mathbf{G}(\tilde{\mathbf{x}}^k - \mathbf{x}^k), \mathbf{x} - \tilde{\mathbf{x}}^k\right\rangle \geq 0,$$

$$h_2(\mathbf{z}) - h_2(\tilde{\mathbf{z}}^k) + \left\langle \rho\mathbf{B}^T\left(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\tilde{\mathbf{z}}^k - \mathbf{c} + \frac{1}{\rho}\mathbf{y}^k\right) + \mathbf{Q}(\tilde{\mathbf{z}}^k - \mathbf{z}^k), \mathbf{z} - \tilde{\mathbf{z}}^k\right\rangle \geq 0.$$

Using the definition of $\tilde{\mathbf{y}}^k$, the above two inequalities can be rewritten as

$$h_1(\mathbf{x}) - h_1(\tilde{\mathbf{x}}^k) + \langle \mathbf{A}^T\tilde{\mathbf{y}}^k + \mathbf{G}(\tilde{\mathbf{x}}^k - \mathbf{x}^k), \mathbf{x} - \tilde{\mathbf{x}}^k\rangle \geq 0,$$

$$h_2(\mathbf{z}) - h_2(\tilde{\mathbf{z}}^k) + \langle \mathbf{B}^T\tilde{\mathbf{y}}^k + (\rho\mathbf{B}^T\mathbf{B} + \mathbf{Q})(\tilde{\mathbf{z}}^k - \mathbf{z}^k), \mathbf{z} - \tilde{\mathbf{z}}^k\rangle \geq 0.$$

Adding the above two inequalities and using the identity

$$\mathbf{y}^{k+1} - \mathbf{y}^k = \rho(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\tilde{\mathbf{z}}^k - \mathbf{c}),$$

we can conclude that for any $\mathbf{x} \in \mathrm{dom}(h_1)$, $\mathbf{z} \in \mathrm{dom}(h_2)$, and $\mathbf{y} \in \mathbb{R}^m$,

$$\boxed{H(\mathbf{x},\mathbf{z}) - H(\tilde{\mathbf{x}}^k, \tilde{\mathbf{z}}^k) + \left\langle \begin{pmatrix} \mathbf{x} - \tilde{\mathbf{x}}^k \\ \mathbf{z} - \tilde{\mathbf{z}}^k \\ \mathbf{y} - \tilde{\mathbf{y}}^k \end{pmatrix}, \begin{pmatrix} \mathbf{A}^T\tilde{\mathbf{y}}^k \\ \mathbf{B}^T\tilde{\mathbf{y}}^k \\ -\mathbf{A}\tilde{\mathbf{x}}^k - \mathbf{B}\tilde{\mathbf{z}}^k + \mathbf{c} \end{pmatrix} - \begin{pmatrix} \mathbf{G}(\mathbf{x}^k - \tilde{\mathbf{x}}^k) \\ \mathbf{C}(\mathbf{z}^k - \tilde{\mathbf{z}}^k) \\ \frac{1}{\rho}(\mathbf{y}^k - \mathbf{y}^{k+1}) \end{pmatrix} \right\rangle \geq 0,} \tag{15.17}$$

where $\mathbf{C} = \rho\mathbf{B}^T\mathbf{B} + \mathbf{Q}$. We will use the following identity that holds for any positive semidefinite matrix $\mathbf{P}$:

$$\boxed{(\mathbf{a} - \mathbf{b})^T\mathbf{P}(\mathbf{c} - \mathbf{d}) = \frac{1}{2}\left(\|\mathbf{a} - \mathbf{d}\|_{\mathbf{P}}^2 - \|\mathbf{a} - \mathbf{c}\|_{\mathbf{P}}^2 + \|\mathbf{b} - \mathbf{c}\|_{\mathbf{P}}^2 - \|\mathbf{b} - \mathbf{d}\|_{\mathbf{P}}^2\right).}$$

Using the above identity, we can conclude that

$$\begin{aligned}(\mathbf{x} - \tilde{\mathbf{x}}^k)^T\mathbf{G}(\mathbf{x}^k - \tilde{\mathbf{x}}^k) &= \frac{1}{2}\left(\|\mathbf{x} - \tilde{\mathbf{x}}^k\|_{\mathbf{G}}^2 - \|\mathbf{x} - \mathbf{x}^k\|_{\mathbf{G}}^2 + \|\tilde{\mathbf{x}}^k - \mathbf{x}^k\|_{\mathbf{G}}^2\right)\\ &\geq \frac{1}{2}\|\mathbf{x} - \tilde{\mathbf{x}}^k\|_{\mathbf{G}}^2 - \frac{1}{2}\|\mathbf{x} - \mathbf{x}^k\|_{\mathbf{G}}^2,\end{aligned} \tag{15.18}$$

as well as

$$(\mathbf{z} - \tilde{\mathbf{z}}^k)^T\mathbf{C}(\mathbf{z}^k - \tilde{\mathbf{z}}^k) = \frac{1}{2}\|\mathbf{z} - \tilde{\mathbf{z}}^k\|_{\mathbf{C}}^2 - \frac{1}{2}\|\mathbf{z} - \mathbf{z}^k\|_{\mathbf{C}}^2 + \frac{1}{2}\|\mathbf{z}^k - \tilde{\mathbf{z}}^k\|_{\mathbf{C}}^2 \tag{15.19}$$

and

$$\begin{aligned}&2(\mathbf{y} - \tilde{\mathbf{y}}^k)^T(\mathbf{y}^k - \mathbf{y}^{k+1})\\ &= \|\mathbf{y} - \mathbf{y}^{k+1}\|^2 - \|\mathbf{y} - \mathbf{y}^k\|^2 + \|\tilde{\mathbf{y}}^k - \mathbf{y}^k\|^2 - \|\tilde{\mathbf{y}}^k - \mathbf{y}^{k+1}\|^2\\ &= \|\mathbf{y} - \mathbf{y}^{k+1}\|^2 - \|\mathbf{y} - \mathbf{y}^k\|^2 + \rho^2\|\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c}\|^2\\ &-\|\mathbf{y}^k + \rho(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c}) - \mathbf{y}^k - \rho(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\tilde{\mathbf{z}}^k - \mathbf{c})\|^2\\ &= \|\mathbf{y} - \mathbf{y}^{k+1}\|^2 - \|\mathbf{y} - \mathbf{y}^k\|^2 + \rho^2\|\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c}\|^2 - \rho^2\|\mathbf{B}(\mathbf{z}^k - \tilde{\mathbf{z}}^k)\|^2.\end{aligned}$$

3.1 Convergence in a strictly contraction sense

定理 2 *Let* $\{v^k\}$ *be the sequence generated by a method for the problem* (1.1) *and* $\tilde{w}^k$ *is obtained in the* k*-th iteration. If* v^k, v^{k+1} *and* $\tilde{w}^k$ *satisfy the conditions in the unified framework, then we have*

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \|v^k - \tilde{v}^k\|_G^2, \quad \forall v^* \in \mathcal{V}^*. \tag{3.5}$$

Proof. Setting $w = w^*$ in (3.1), we get

$$\begin{aligned} & \|v^k - v^*\|_H^2 - \|v^{k+1} - v^*\|_H^2 \\ \ge \quad & \|v^k - \tilde{v}^k\|_G^2 + 2\{\theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(\tilde{w}^k)\}. \end{aligned} \tag{3.6}$$

By using the optimality of w^* and the monotonicity of $F(w)$, we have

$$\theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(\tilde{w}^k) \ge \theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(w^*) \ge 0$$

and thus

$$\|v^k - v^*\|_H^2 - \|v^{k+1} - v^*\|_H^2 \ge \|v^k - \tilde{v}^k\|_G^2. \tag{3.7}$$

The assertion (3.5) follows directly. $\square$

定理 1 中的结论 (3.1)

$$\begin{aligned} & \theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^T F(\tilde{w}^k) \\ \geq \quad & \frac{1}{2}\left(\|v-v^{k+1}\|_H^2-\|v-v^k\|_H^2\right)+\frac{1}{2}\|v^k-\tilde{v}^k\|_G^2, \quad \forall w \in \Omega. \end{aligned}$$

是为收敛速率的证明而准备的.

否则, 我们可以通过在 (3.2)

$$\theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^T F(\tilde{w}^k) \geq (v-\tilde{v}^k)^T H(v^k-v^{k+1}), \ \forall w \in \Omega.$$

中令 $w=w^*$, 得到

$$(v^k-v^{k+1})^T H(\tilde{v}^k-v^*) \geq 0. \tag{3.8}$$

将恒等式

$$(a-b)^T H(c-d)=\frac{1}{2}\{\|a-d\|_H^2-\|b-d\|_H^2\}-\frac{1}{2}\{\|a-c\|_H^2-\|b-c\|_H^2\}$$

用于 (3.8) 的左端, 令 $a=v^k, \ b=v^{k+1}, \ c=\tilde{v}^k$ 和 $d=v^*$, 我们得到

$$\begin{aligned} & (v^k-v^{k+1})^T H(\tilde{v}^k-v^*) \\ = \quad & \frac{1}{2}\{\|v^k-v^*\|_H^2-\|v^{k+1}-v^*\|_H^2\}-\frac{1}{2}\{\|v^k-\tilde{v}^k\|_H^2-\|v^{k+1}-\tilde{v}^k\|_H^2\}. \end{aligned}$$

根据 (3.8) 就有

$$\|v^k - v^*\|_H^2 - \|v^{k+1} - v^*\|_H^2 \geq \|v^k - \tilde{v}^k\|_H^2 - \|v^{k+1} - \tilde{v}^k\|_H^2. \tag{3.9}$$

再把上式的右端化简一下,

$$\begin{aligned}
&\|v^k - \tilde{v}^k\|_H^2 - \|v^{k+1} - \tilde{v}^k\|_H^2 \\
&\quad = \quad \|v^k - \tilde{v}^k\|_H^2 - \|(v^k - \tilde{v}^k) - (v^k - v^{k+1})\|_H^2 \\
&\quad \stackrel{(1.2b)}{=} \quad \|v^k - \tilde{v}^k\|_H^2 - \|(v^k - \tilde{v}^k) - M(v^k - \tilde{v}^k)\|_H^2 \\
&\quad = \quad 2(v^k - \tilde{v}^k)^T HM(v^k - \tilde{v}^k) - (v^k - \tilde{v}^k)^T M^T HM(v^k - \tilde{v}^k) \\
&\quad = \quad (v^k - \tilde{v}^k)^T (Q^T + Q - M^T HM)(v^k - \tilde{v}^k) \\
&\quad \stackrel{(1.3b)}{=} \quad \|v^k - \tilde{v}^k\|_G^2.
\end{aligned} \tag{3.10}$$

将 (3.10) 代入 (3.9) 就得到引理的结论. $\square$

3.2 Convergence rate （两篇主要理论文章）

Convergence rate in an ergodic sense [10]

为了证明算法遍历意义下的迭代复杂性, 我们需要对变分不等式 (1.1) 的解集做新的刻

画. 由于 (1.1) 中的仿射算子 F 恰有

$$(w-w^*)^T F(w^*) = (w-w^*)^T F(w),$$

变分不等式问题

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w-w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega,$$

和

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w-w^*)^T F(w) \ge 0, \quad \forall\, w \in \Omega$$

是等价的. 我们用后者定义变分不等式 (1.1) 的近似解. 对给定的 $\epsilon > 0$, 如果 $\tilde{w}$ 满足

$$\tilde{w} \in \Omega, \quad \theta(u) - \theta(\tilde{u}) + (w-\tilde{w})^T F(w) \ge -\epsilon, \quad \forall\; w \in \mathcal{D}_{(\tilde{w})}, \tag{3.11a}$$

其中

$$\mathcal{D}_{(\tilde{w})} = \{w \in \Omega \,|\, \|w-\tilde{w}\| \le 1\}, \tag{3.11b}$$

就叫做变分不等式 (1.1) 的 ϵ 近似解. 它可以等价地表示成

$$\tilde{w} \in \Omega, \quad \sup_{w \in \mathcal{D}_{(\tilde{w})}} \left\{\theta(\tilde{u}) - \theta(u) + (\tilde{w}-w)^T F(w)\right\} \le \epsilon. \tag{3.12}$$

人们感兴趣的是：对给定的 $\epsilon > 0$, 经过多少次迭代得到一个 $\tilde{w} \in \Omega$, 使得 (3.12) 成立.

这就是我们要讨论的遍历意义下的收敛速率. 讨论遍历意义下的收敛性, 对 (1.3) 中的矩阵 H 和 G, 只要求它半正定.

Theorem 1 is also the base for the convergence rate proof. Using the monotonicity of F, we have

$$(w-\tilde{w}^k)^T F(w) = (w-\tilde{w}^k)^T F(\tilde{w}^k).$$

Substituting it in (3.1), we obtain

$$\theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^T F(w)+\frac{1}{2}\|v-v^k\|_H^2 \geq \frac{1}{2}\|v-v^{k+1}\|_H^2, \ \forall w\in\Omega. \tag{3.13}$$

Note that the above assertion is hold for $G \succeq 0$.

定理 3 *Let $\{v^k\}$ be the sequence generated by a method for the problem* (1.1) *and $\tilde{w}^k$ is obtained in the k-th iteration. Assume that v^k, v^{k+1} and $\tilde{w}^k$ satisfy the conditions in the unified framework and let $\tilde{w}_t$ be defined by*

$$\tilde{w}_t = \frac{1}{t+1}\sum_{k=0}^{t}\tilde{w}^k. \tag{3.14}$$

Then, for any integer number $t>0$, $\tilde{w}_t\in\Omega$ and

$$\theta(\tilde{u}_t)-\theta(u)+(\tilde{w}_t-w)^T F(w) \leq \frac{1}{2(t+1)}\|v-v^0\|_H^2, \quad \forall w\in\Omega. \tag{3.15}$$

Proof. First, it holds that $\tilde{w}^k \in \Omega$ for all $k \geq 0$. Together with the convexity of Ω, (3.14) implies that $\tilde{w}_t \in \Omega$. Rewriting the inequality (3.13) in its equivalent form

$$\theta(\tilde{u}^k) - \theta(u) + (\tilde{w}^k - w)^T F(w) + \frac{1}{2}\|v - v^{k+1}\|_H^2 \leq \frac{1}{2}\|v - v^k\|_H^2, \ \forall w \in \Omega.$$

Summing the last inequality over $k = 0, 1, \ldots, t$, we obtain

$$\sum_{k=0}^{t} \theta(\tilde{u}^k) - (t+1)\theta(u) + \Big(\sum_{k=0}^{t} \tilde{w}^k - (t+1)w\Big)^T F(w) \leq \frac{1}{2}\|v - v^0\|_H^2, \quad \forall w \in \Omega.$$

Use the notation of $\tilde{w}_t$, it can be written as

$$\frac{1}{t+1}\sum_{k=0}^{t} \theta(\tilde{u}^k) - \theta(u) + (\tilde{w}_t - w)^T F(w) \leq \frac{1}{2\alpha(t+1)}\|v - v^0\|_H^2, \quad \forall w \in \Omega. \tag{3.16}$$

Since $\theta(u)$ is convex and

$$\tilde{u}_t = \frac{1}{t+1}\sum_{k=0}^{t} \tilde{u}^k,$$

we have

$$\theta(\tilde{u}_t) \le \frac{1}{t+1} \sum_{k=0}^{t} \theta(\tilde{u}^k).$$

Substituting it in (3.16), the assertion of this theorem follows directly. $\square$

Recall (3.12). The conclusion (3.15) thus indicates obviously that the method is able to generate an approximate solution (i.e., $\tilde{w}_t$) with the accuracy $O(1/t)$ after t iterations. That is, in the case $G \succeq 0$, the convergence rate $O(1/t)$ of the method is established.

我们 2012 年发表在 SIAM Numerical Analysis 的论文 [10] 就是用这种方式证明了交替方向法在遍历意义下 $O(1/t)$ 的收敛速率. 这被认为我们在交替方向法方面的一个比较重要的结果, 这里只是说明, 该结果对符合统一框架收敛条件的方法都是成立的.

Convergence rate in a pointwise iteration-complexity [12]

我们 2015 年发表在 Numerische Mathematik 的论文 [12] 证明了交替方向法在点列意义下一个比较重要的结果.

$$\|v^{k+1} - v^{k+2}\|_H \le \|v^k - v^{k+1}\|_H.$$

这个性质已经被一些学者用来研发加速 ADMM. 下面证明这个结果对符合统一框架收敛条件的方法也都成立. 证明也只需要矩阵 H 和 G 半正定.

定理 4 *For solving the variational inequality* (1.1)*, let* $\{w^k\}, \{\tilde{w}^k\}$ *be the sequence generated by* (1.2)*. If the conditions* (1.3) *are satisfied, then we have*

$$\|v^{k+1} - v^{k+2}\|_H^2 \le \|v^k - v^{k+1}\|_H^2 - \|(v^k - \tilde{v}^k) - (v^{k+1} - \tilde{v}^{k+1})\|_G^2. \quad (3.17)$$

Proof Note that we have

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \ \forall w \in \Omega$$

and

$$\theta(u) - \theta(\tilde{u}^{k+1}) + (w - \tilde{w}^{k+1})^T F(\tilde{w}^{k+1}) \ge (v - \tilde{v}^{k+1})^T Q(v^{k+1} - \tilde{v}^{k+1}), \ \forall w \in \Omega.$$

Set the vector w in the above two inequalities by $\tilde{w}^{k+1}$ and $\tilde{w}^k$, respectively, we get

$$\theta(\tilde{u}^{k+1}) - \theta(\tilde{u}^k) + (\tilde{w}^{k+1} - \tilde{w}^k)^T F(\tilde{w}^k) \ge (\tilde{v}^{k+1} - \tilde{v}^k)^T Q(v^k - \tilde{v}^k)$$

and

$$\theta(\tilde{u}^k) - \theta(\tilde{u}^{k+1}) + (\tilde{w}^k - \tilde{w}^{k+1})^T F(\tilde{w}^{k+1}) \ge (\tilde{v}^k - \tilde{v}^{k+1})^T Q(v^{k+1} - \tilde{v}^{k+1}).$$

Adding the above two inequalities, it follows that

$$(\tilde{v}^k - \tilde{v}^{k+1})^T Q\{(v^k - \tilde{v}^k) - (v^{k+1} - \tilde{v}^{k+1})\} \ge 0.$$

Adding $\{(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\}^T Q\{(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\}$ to the both sides of the last inequality, we get

$$(v^k - v^{k+1})^T Q\{(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\} \ge \frac{1}{2}\|(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\|^2_{(Q^T+Q)},$$

and thus

$$(v^k - v^{k+1})^T H\{(v^k - v^{k+1}) - (v^{k+1} - v^{k+2})\} \ge \frac{1}{2}\|(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\|^2_{(Q^T+Q)}. \tag{3.18}$$

Finally, by using $\|a\|_H^2 - \|b\|_H^2 = 2a^T H(a-b) - \|a-b\|_H^2$ and (3.18), we get

$$\begin{aligned}
&\|v^k - v^{k+1}\|_H^2 - \|v^{k+1} - v^{k+2}\|_H^2 \\
&\quad = \; 2(v^k - v^{k+1})^T H\{(v^k - v^{k+1}) - (v^{k+1} - v^{k+2})\} \\
&\qquad\quad - \|(v^k - v^{k+1}) - (v^{k+1} - v^{k+2})\|_H^2 \\
&\quad \ge \; \|(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\|^2_{(Q^T+Q)} - \|(v^k - v^{k+1}) - (v^{k+1} - v^{k+2})\|_H^2 \\
&\quad = \; \|(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\|^2_{(Q^T+Q-M^THM)} \\
&\quad = \; \|(v^k - \tilde v^k) - (v^{k+1} - \tilde v^{k+1})\|^2_G.
\end{aligned}$$

This is the equivalent form of (3.17) and the proof is complete. $\square$

4 ADMM for problems with two separable blocks

This section concern the structured convex optimization problem namely,

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\, x \in \mathcal{X}, y \in \mathcal{Y}\}. \tag{4.1}$$

The Lagrangian function and the augmented Lagrange Function of (4.1) are

$$L^{[2]}(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b).$$

and

$$\mathcal{L}_\beta^{[2]}(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b) + \frac{\beta}{2}\|Ax + By - b\|^2, \tag{4.2}$$

respectively. Recall the model (4.1) can be explained as the VI

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{4.3a}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y), \quad w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \tag{4.3b}$$

$$F(w) = \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax + By - b \end{pmatrix}, \quad \text{and} \quad \Omega = \mathcal{X} \times \mathcal{Y} \times \Re^m. \tag{4.3c}$$

Using the augmented Lagrange function, the recursion of the alternating direction method of multipliers for the structured convex optimization (4.1) can be written as

$$\begin{cases} x^{k+1} \in \text{Argmin}\{\mathcal{L}_\beta^{[2]}(x, y^k, \lambda^k) \,|\, x \in \mathcal{X}\}, \\ y^{k+1} \in \text{Argmin}\{\mathcal{L}_\beta^{[2]}(x^{k+1}, y, \lambda^k) \,|\, y \in \mathcal{Y}\}, \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). \end{cases} \tag{4.4}$$

Note that the essential variable of ADMM (4.4) is $v = (y, \lambda)$.

统一框架下的 ADMM. ADMM scheme (4.4) is also a special case which belongs to the unified algorithmic framework (1.2) and the Convergence Condition is satisfied.

In order to cast the ADMM scheme (4.4) into a special case of (1.2), let us first define the artificial vector $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ by

$$\tilde{x}^k = x^{k+1}, \quad \tilde{y}^k = y^{k+1} \quad \text{and} \quad \tilde{\lambda}^k = \lambda^k - \beta(Ax^{k+1} + By^k - b), \tag{4.5}$$

where (x^{k+1}, y^{k+1}) is generated by the ADMM (4.4).

我们注意到 A. Beck 在他的专著 First-Order Methods in convex optimization [1], 也采用了这种转换.

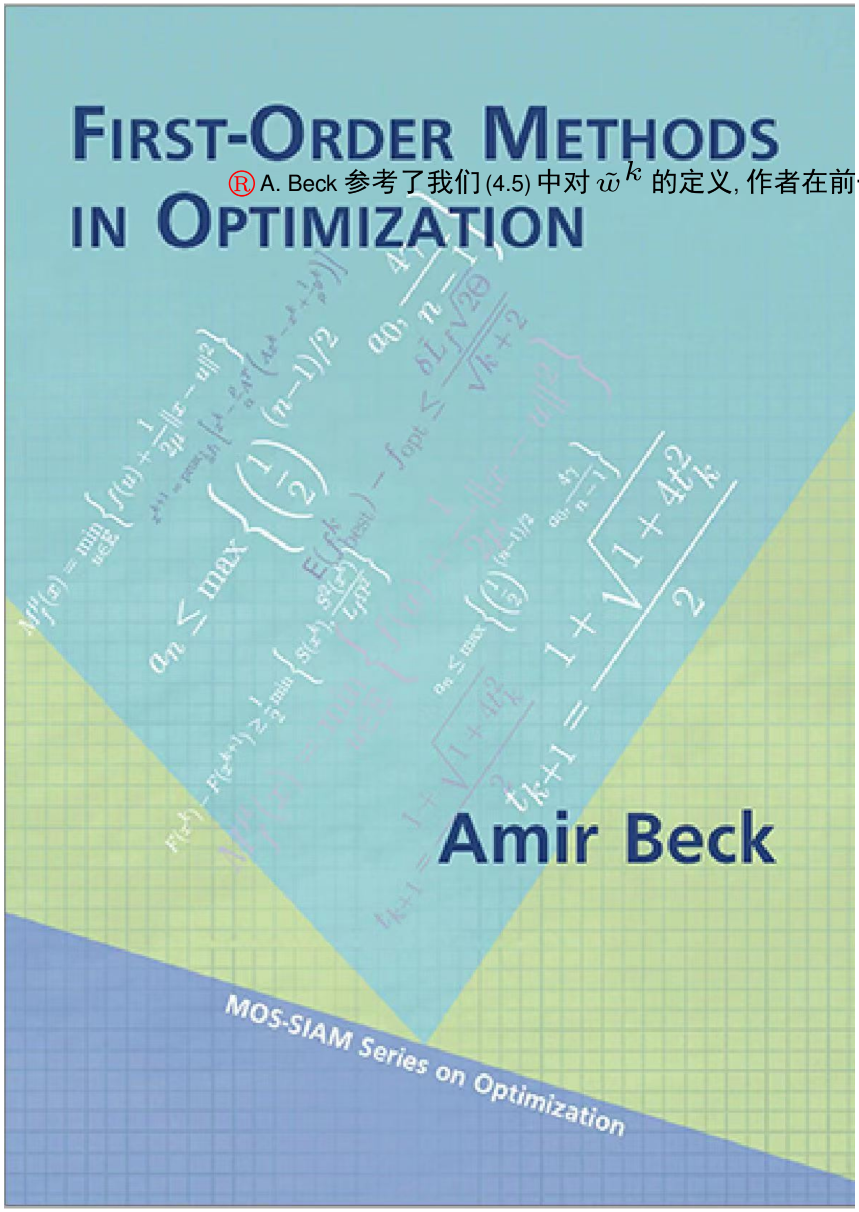
Prediction

$$\left\{\begin{array}{l} \tilde{x}^k \in \text{Argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{\beta}{2}\|Ax + By^k - b\|^2 \,|\, x \in \mathcal{X}\}, \\ \tilde{y}^k \in \text{Argmin}\{\theta_2(y) - y^T B^T \lambda^k + \frac{\beta}{2}\|A\tilde{x}^k + By - b\|^2 \,|\, y \in \mathcal{Y}\}, \\ \tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b). \end{array}\right. \tag{4.6}$$

According to the scheme (4.4), the defined artificial vector $\tilde{w}^k$ satisfies the following VI: $\tilde{w}^k \in \Omega$,

$$\left\{\begin{array}{ll} \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{-A^T\tilde{\lambda}^k\} \ge 0, & \forall\, x \in \mathcal{X}, \\ \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T\{-B^T\tilde{\lambda}^k + \beta B^T B(\tilde{y}^k - y^k)\} \ge 0, & \forall\, y \in \mathcal{Y}, \\ \qquad (A\tilde{x}^k + B\tilde{y}^k - b) - B(\tilde{y}^k - y^k) + \dfrac{1}{\beta}(\tilde{\lambda}^k - \lambda^k) = 0. & \end{array}\right.$$

This can be written in form of (1.2a) as described in the following lemma.



Ⓡ A. Beck 参考了我们 (4.5) 中对 $\tilde{w}^k$ 的定义, 作者在前一页的脚注做了说明

We will use the following notation:

$$\boxed{\begin{aligned}\tilde{\mathbf{x}}^k &= \mathbf{x}^{k+1},\\ \tilde{\mathbf{z}}^k &= \mathbf{z}^{k+1},\\ \tilde{\mathbf{y}}^k &= \mathbf{y}^k + \rho(\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{z}^k - \mathbf{c}).\end{aligned}}$$

Using (15.15), (15.16), the subgradient inequality, and the above notation, we obtain that for any $\mathbf{x} \in \operatorname{dom}(h_1)$ and $\mathbf{z} \in \operatorname{dom}(h_2)$,

$$h_1(\mathbf{x}) - h_1(\tilde{\mathbf{x}}^k) + \left\langle \rho\mathbf{A}^T\left(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c} + \frac{1}{\rho}\mathbf{y}^k\right) + \mathbf{G}(\tilde{\mathbf{x}}^k - \mathbf{x}^k), \mathbf{x} - \tilde{\mathbf{x}}^k\right\rangle \geq 0,$$

$$h_2(\mathbf{z}) - h_2(\tilde{\mathbf{z}}^k) + \left\langle \rho\mathbf{B}^T\left(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\tilde{\mathbf{z}}^k - \mathbf{c} + \frac{1}{\rho}\mathbf{y}^k\right) + \mathbf{Q}(\tilde{\mathbf{z}}^k - \mathbf{z}^k), \mathbf{z} - \tilde{\mathbf{z}}^k\right\rangle \geq 0.$$

Using the definition of $\tilde{\mathbf{y}}^k$, the above two inequalities can be rewritten as

$$\begin{aligned}h_1(\mathbf{x}) - h_1(\tilde{\mathbf{x}}^k) + \langle \mathbf{A}^T\tilde{\mathbf{y}}^k + \mathbf{G}(\tilde{\mathbf{x}}^k - \mathbf{x}^k), \mathbf{x} - \tilde{\mathbf{x}}^k\rangle &\geq 0,\\ h_2(\mathbf{z}) - h_2(\tilde{\mathbf{z}}^k) + \langle \mathbf{B}^T\tilde{\mathbf{y}}^k + (\rho\mathbf{B}^T\mathbf{B} + \mathbf{Q})(\tilde{\mathbf{z}}^k - \mathbf{z}^k), \mathbf{z} - \tilde{\mathbf{z}}^k\rangle &\geq 0.\end{aligned}$$

Adding the above two inequalities and using the identity

$$\mathbf{y}^{k+1} - \mathbf{y}^k = \rho(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\tilde{\mathbf{z}}^k - \mathbf{c}),$$

we can conclude that for any $\mathbf{x} \in \operatorname{dom}(h_1)$, $\mathbf{z} \in \operatorname{dom}(h_2)$, and $\mathbf{y} \in \mathbb{R}^m$,

$$\boxed{H(\mathbf{x}, \mathbf{z}) - H(\tilde{\mathbf{x}}^k, \tilde{\mathbf{z}}^k) + \left\langle \begin{pmatrix}\mathbf{x} - \tilde{\mathbf{x}}^k\\ \mathbf{z} - \tilde{\mathbf{z}}^k\\ \mathbf{y} - \tilde{\mathbf{y}}^k\end{pmatrix}, \begin{pmatrix}\mathbf{A}^T\tilde{\mathbf{y}}^k\\ \mathbf{B}^T\tilde{\mathbf{y}}^k\\ -\mathbf{A}\tilde{\mathbf{x}}^k - \mathbf{B}\tilde{\mathbf{z}}^k + \mathbf{c}\end{pmatrix} - \begin{pmatrix}\mathbf{G}(\mathbf{x}^k - \tilde{\mathbf{x}}^k)\\ \mathbf{C}(\mathbf{z}^k - \tilde{\mathbf{z}}^k)\\ \frac{1}{\rho}(\mathbf{y}^k - \mathbf{y}^{k+1})\end{pmatrix}\right\rangle \geq 0,} \tag{15.17}$$

where $\mathbf{C} = \rho\mathbf{B}^T\mathbf{B} + \mathbf{Q}$. We will use the following identity that holds for any positive semidefinite matrix $\mathbf{P}$:

$$\boxed{(\mathbf{a} - \mathbf{b})^T\mathbf{P}(\mathbf{c} - \mathbf{d}) = \frac{1}{2}\left(\|\mathbf{a} - \mathbf{d}\|_{\mathbf{P}}^2 - \|\mathbf{a} - \mathbf{c}\|_{\mathbf{P}}^2 + \|\mathbf{b} - \mathbf{c}\|_{\mathbf{P}}^2 - \|\mathbf{b} - \mathbf{d}\|_{\mathbf{P}}^2\right).}$$

Using the above identity, we can conclude that

$$\begin{aligned}(\mathbf{x} - \tilde{\mathbf{x}}^k)^T\mathbf{G}(\mathbf{x}^k - \tilde{\mathbf{x}}^k) &= \frac{1}{2}\left(\|\mathbf{x} - \tilde{\mathbf{x}}^k\|_{\mathbf{G}}^2 - \|\mathbf{x} - \mathbf{x}^k\|_{\mathbf{G}}^2 + \|\tilde{\mathbf{x}}^k - \mathbf{x}^k\|_{\mathbf{G}}^2\right)\\ &\geq \frac{1}{2}\|\mathbf{x} - \tilde{\mathbf{x}}^k\|_{\mathbf{G}}^2 - \frac{1}{2}\|\mathbf{x} - \mathbf{x}^k\|_{\mathbf{G}}^2,\end{aligned} \tag{15.18}$$

as well as

$$(\mathbf{z} - \tilde{\mathbf{z}}^k)^T\mathbf{C}(\mathbf{z}^k - \tilde{\mathbf{z}}^k) = \frac{1}{2}\|\mathbf{z} - \tilde{\mathbf{z}}^k\|_{\mathbf{C}}^2 - \frac{1}{2}\|\mathbf{z} - \mathbf{z}^k\|_{\mathbf{C}}^2 + \frac{1}{2}\|\mathbf{z}^k - \tilde{\mathbf{z}}^k\|_{\mathbf{C}}^2 \tag{15.19}$$

and

$$\begin{aligned}&2(\mathbf{y} - \tilde{\mathbf{y}}^k)^T(\mathbf{y}^k - \mathbf{y}^{k+1})\\ &= \|\mathbf{y} - \mathbf{y}^{k+1}\|^2 - \|\mathbf{y} - \mathbf{y}^k\|^2 + \|\tilde{\mathbf{y}}^k - \mathbf{y}^k\|^2 - \|\tilde{\mathbf{y}}^k - \mathbf{y}^{k+1}\|^2\\ &= \|\mathbf{y} - \mathbf{y}^{k+1}\|^2 - \|\mathbf{y} - \mathbf{y}^k\|^2 + \rho^2\|\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c}\|^2\\ &- \|\mathbf{y}^k + \rho(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c}) - \mathbf{y}^k - \rho(\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\tilde{\mathbf{z}}^k - \mathbf{c})\|^2\\ &= \|\mathbf{y} - \mathbf{y}^{k+1}\|^2 - \|\mathbf{y} - \mathbf{y}^k\|^2 + \rho^2\|\mathbf{A}\tilde{\mathbf{x}}^k + \mathbf{B}\mathbf{z}^k - \mathbf{c}\|^2 - \rho^2\|\mathbf{B}(\mathbf{z}^k - \tilde{\mathbf{z}}^k)\|^2.\end{aligned}$$

引理 1 *For given* v^k, *let* w^{k+1} *be generated by* (4.4) *and* $\tilde{w}^k$ *be defined by* (4.5). *Then, we have*

$$\tilde{w}^k \in \Omega,\ \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k),\ \forall w \in \Omega,$$

where

$$Q = \begin{pmatrix} \beta B^T B & 0 \\ -B & \frac{1}{\beta} I \end{pmatrix}. \tag{4.7}$$

Recall the essential variable of the ADMM scheme (4.4) is (y, λ). Moreover, using the definition of $\tilde{w}^k$, the λ^{k+1} updated by (4.4) can be represented as

$$\begin{aligned} \lambda^{k+1} &= \lambda^k - \beta(A\tilde{x}^k + B\tilde{y}^k - b) \\ &= \lambda^k - \big[-\beta B(y^k - \tilde{y}^k) + \beta(A\tilde{x}^k + By^k - b)\big] \\ &= \lambda^k - \big[-\beta B(y^k - \tilde{y}^k) + (\lambda^k - \tilde{\lambda}^k)\big]. \end{aligned}$$

Therefore, the ADMM scheme (4.4) can be written as

$$\begin{pmatrix} y^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} y^k \\ \lambda^k \end{pmatrix} - \begin{pmatrix} I & 0 \\ -\beta B & I \end{pmatrix} \begin{pmatrix} y^k - \tilde{y}^k \\ \lambda^k - \tilde{\lambda}^k \end{pmatrix}. \tag{4.8a}$$

which corresponds to the step (1.2b) with

$$M = \begin{pmatrix} I & 0 \\ -\beta B & I \end{pmatrix} \quad \text{and} \quad \alpha = 1. \tag{4.8b}$$

$\boxed{\text{验证收敛性条件.}}$ Now we check that the Convergence Condition is satisfied by the ADMM scheme (4.4). Indeed, for the matrix M in (4.8b), we have

$$M^{-1} = \begin{pmatrix} I & 0 \\ \beta B & I \end{pmatrix}.$$

Thus, by using (4.7) and (4.8b), we obtain $\boxed{\text{验证 } H \text{ 半正定}}$

$$H = QM^{-1} = \begin{pmatrix} \beta B^T B & 0 \\ -B & \frac{1}{\beta} I \end{pmatrix} \begin{pmatrix} I & 0 \\ \beta B & I \end{pmatrix} = \begin{pmatrix} \beta B^T B & 0 \\ 0 & \frac{1}{\beta} I \end{pmatrix}, \tag{4.9}$$

and consequently

验证 G 的半正定

$$
\begin{aligned}
G &= Q^T + Q - \alpha M^T H M = Q^T + Q - Q^T M \\
&= \begin{pmatrix} 2\beta B^T B & -B^T \\ -B & \frac{2}{\beta} I \end{pmatrix} - \begin{pmatrix} \beta B^T B & -B^T \\ 0 & \frac{1}{\beta} I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\beta B & I \end{pmatrix} \\
&= \begin{pmatrix} 2\beta B^T B & -B^T \\ -B & \frac{2}{\beta} I \end{pmatrix} - \begin{pmatrix} 2\beta B^T B & -B^T \\ -B & \frac{1}{\beta} I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\beta} I \end{pmatrix}. \quad (4.10)
\end{aligned}
$$

Therefore, H is symmetric and positive definite under the assumption that B is full column rank; and G is positive semi-definite. The Convergence Condition is satisfied; and thus the convergence of the ADMM scheme (4.4) is guaranteed.

见论文 [10]

收缩性和点列意义下的收敛速率. 我们将经典的 ADMM 按统一框架故意解释成预测-校正方法. 经过 (4.6) 预测以后, 再由

$$
v^{k+1} = v^k - M(v^k - \tilde{v}^k) \quad (4.11)
$$

校正. 这符合统一框架的模式 (1.2). 在 (4.9) 和 (4.10) 中我们分别验证了矩阵 H 和 G 是

半正定的. 因此, 根据定理 4 就有

$$\|v^{k+1}-v^{k+2}\|_H \le \|v^k-v^{k+1}\|_H, \quad \forall\, k>0. \tag{4.12}$$

由 (4.12), 对任意的正整数 $t>0$,

$$\begin{aligned}
\|v^t-v^{t+1}\|_H^2 \quad &\le \quad \frac{1}{t+1}\sum_{k=0}^{t}\|v^k-v^{k+1}\|_H^2 \\
&\le \quad \frac{1}{t+1}\sum_{k=0}^{\infty}\|v^k-v^{k+1}\|_H^2 \\
&\overset{(4.12)}{\le} \quad \frac{1}{t+1}\|v^0-v^*\|_H^2.
\end{aligned}$$

人们往往用 $\|v^t-v^{t+1}\|_H^2$ 的大小做停机准则的参考. 见论文 [12]

5 利用统一框架验证 ADMM 类算法的收敛性

5.1 交换顺序的交替方向法

将经典的 ADMM (4.4) 中求解 y-子问题和校正 λ 的顺序交换, 通过

$$\left\{\begin{array}{lcl} x^{k+1} & \in & \text{Argmin}\{\mathcal{L}_{\beta}^{[2]}(x, y^k, \lambda^k) \,|\, x \in \mathcal{X}\}, \\ \lambda^{k+1} & = & \lambda^k - \beta(Ax^{k+1} + By^k - b), \\ y^{k+1} & \in & \text{Argmin}\{\mathcal{L}_{\beta}^{[2]}(x^{k+1}, y, \lambda^{k+1}) \,|\, y \in \mathcal{Y}\}, \end{array}\right. \tag{5.1}$$

得到的 $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ 作为预测点, 然后

$$\left\{\begin{array}{lcl} y^{k+1} & := & y^k - \gamma(y^k - y^{k+1}), \\ \lambda^{k+1} & := & \lambda^k - \gamma(\lambda^k - \lambda^{k+1}). \end{array}\right. \quad \text{（松弛延拓）} \tag{5.2}$$

这里 $\gamma \in (0, 2)$. 赋值号 “:=” 表示 (5.2) 右端的 (y^{k+1}, λ^{k+1}) 是由算法的前半部分 (5.1) 产生的. (5.2) 左端才是下一步迭代开始所需要的 (y^{k+1}, λ^{k+1}). 对多数问题, 这样往往能加快收敛.

注意到, (5.1) 中核心变量还是 $v = (y, \lambda)$. 先把 (5.1) 产生的 w^{k+1} 本身定义成预测

点 $\tilde{w}^k$, 即

$$\tilde{w}^k = \begin{pmatrix} \tilde{x}^k \\ \tilde{y}^k \\ \tilde{\lambda}^k \end{pmatrix} = \begin{pmatrix} x^{k+1} \\ y^{k+1} \\ \lambda^k - \beta(Ax^{k+1} + By^k - b) \end{pmatrix}, \tag{5.3}$$

利用 $\mathcal{L}_\beta^{[2]}(x,y,\lambda)$ 的表达式, 交换顺序的交替方向法迭代公式 (5.1) 可以表示成

$$\begin{cases} \tilde{x}^k \in \operatorname{argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{\beta}{2}\|Ax + By^k - b\|^2 \mid x \in \mathcal{X}\}, & \text{(5.4a)} \\ \tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b), & \text{(5.4b)} \\ \tilde{y}^k \in \operatorname{argmin}\{\theta_2(y) - y^T B^T \tilde{\lambda}^k + \frac{\beta}{2}\|A\tilde{x}^k + By - b\|^2 \mid y \in \mathcal{Y}\}. & \text{(5.4c)} \end{cases}$$

我们用统一框架来证明交换顺序的交替方向法 (5.1)-(5.2) 的收敛性. 先给出由 (5.4) 求得的 $\tilde{w}^k$ 满足的形如 (1.2a) 的预测公式.

首先, 根据第一讲的定理 1, (5.4a) 的最优性条件是

$$\tilde{x}^k \in \mathcal{X}, \;\; \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{-A^T\lambda^k + \beta A^T(A\tilde{x}^k + By^k - b)\} \ge 0, \; \forall\, x \in \mathcal{X}.$$

利用 $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, 上式就是

$$\theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{\underline{-A^T\tilde{\lambda}^k}\} \ge 0, \;\; \forall\, x \in \mathcal{X}. \tag{5.5a}$$

类似地, 根据第一讲的定理 1, (5.4c) 的最优性条件是

$$\tilde{y}^k \in \mathcal{Y}, \quad \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T \big\{ -B^T\tilde{\lambda}^k + \beta B^T(A\tilde{x}^k + B\tilde{y}^k - b) \big\} \geq 0, \ \forall\, y \in \mathcal{Y}.$$

由于 $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, 我们有

$$\begin{aligned} & -B^T\tilde{\lambda}^k + \beta B^T(A\tilde{x}^k + B\tilde{y}^k - b) \\ = \quad & -B^T\tilde{\lambda}^k + \beta B^T B(\tilde{y}^k - y^k) + \beta B^T(A\tilde{x}^k + By^k - b) \\ = \quad & -B^T\tilde{\lambda}^k + \beta B^T B(\tilde{y}^k - y^k) - B^T(\tilde{\lambda}^k - \lambda^k). \end{aligned}$$

因此, y-子问题 (5.4c) 的最优性条件是

$$\begin{aligned} \tilde{y}^k \in \mathcal{Y}, \ \ \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T \big\{ & \underline{-B^T\tilde{\lambda}^k} + \beta B^T B(\tilde{y}^k - y^k) \\ & - B^T(\tilde{\lambda}^k - \lambda^k) \big\} \geq 0, \quad \forall y \in \mathcal{Y}. \end{aligned} \tag{5.5b}$$

对于 (5.4b) 中给出的 $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, 可以表示成

$$(A\tilde{x}^k + B\tilde{y}^k - b) - B(\tilde{y}^k - y^k) + \frac{1}{\beta}(\tilde{\lambda}^k - \lambda^k) = 0,$$

也就是

$$\tilde{\lambda}^k \in \Re^m, \quad (\lambda - \tilde{\lambda}^k)^T \{ \underline{(A\tilde{x}^k + B\tilde{y}^k - b)} \\ - B(\tilde{y}^k - y^k) + \frac{1}{\beta}(\tilde{\lambda}^k - \lambda^k) \} \geq 0, \quad \forall \lambda \in \Re^m. \tag{5.5c}$$

将 (5.5a), (5.5b) 和 (5.5c) 组合在一起, 注意到下划线部分是 (4.3) 中的 $F(\tilde{w}^k)$, 我们得到下面的引理.

引理 2 求解变分不等式 (4.3), 对给定的 v^k, 由 (5.4) 提供的 $\tilde{w}^k$ 满足

$$\tilde{w}^k \in \Omega, \;\; \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (v - \tilde{v}^k)^T H(v^k - \tilde{v}^k), \;\; \forall w \in \Omega,$$

其中

$$H = \begin{pmatrix} \beta B^T B & -B^T \\ -B & \frac{1}{\beta} I_m \end{pmatrix}. \tag{5.6}$$

这里的矩阵 H 是半正定的, 取 $\alpha = 1$ 的平凡校正, 就是方法 (5.1). 它的收敛效果与经典的 ADMM (4.4) 别无二致. 但是, 如果做平凡的松弛延伸, 让

$$v^{k+1} = v^k - \alpha(v^k - \tilde{v}^k), \quad \alpha = 1.5 \in (0, 2),$$

就相当于方法 (5.1)-(5.1), 收敛速度往往会有 30% 的提高.

从理论上来讲, (5.6) 中的矩阵 H 即使在 B 列满秩的时候也是半正定的, 但这并不影响

计算和收敛性态. 当然, 我们也可以通过在子问题 (5.4c) 的目标函数中增添一项 $\frac{\delta}{2}\|y-y^k\|^2$, 就能使 相应的 H 矩阵变成

$$H=\begin{pmatrix} \beta B^TB+\delta I_{n_2} & -B^T \\ -B & \frac{1}{\beta}I_m \end{pmatrix}.$$

对任意的 $\beta,\delta>0$, 上面的 H 矩阵是正定的.

5.2 对称的交替方向法

人们习惯于用经典的乘子交替方向法 (4.4) 求解问题 (4.1). 从问题 (4.1) 本身看, 原始变量 x 和 y 是平等的, 在算法设计上平等对待 x 和 y 子问题, 也是最自然不过的考虑. 因此我们采用对称的交替方向法 [6], 它的 k步迭代也是从给定的 (y^k,λ^k) 开始, 通过

$$\text{(S-ADMM)}\quad \left\{\begin{array}{ll} x^{k+1}\in \operatorname{argmin}\{\mathcal{L}_\beta^{[2]}(x,y^k,\lambda^k)\,|\,x\in\mathcal{X}\}, & \text{(5.7a)}\\ \lambda^{k+\frac12}=\lambda^k-\mu\beta(Ax^{k+1}+By^k-b), & \text{(5.7b)}\\ y^{k+1}\in \operatorname{argmin}\{\mathcal{L}_\beta^{[2]}(x^{k+1},y,\lambda^{k+\frac12})\,|\,y\in\mathcal{Y}\}, & \text{(5.7c)}\\ \lambda^{k+1}=\lambda^{k+\frac12}-\mu\beta(Ax^{k+1}+By^{k+1}-b). & \text{(5.7d)} \end{array}\right.$$

得到新的迭代点 $w^{k+1}=(x^{k+1},y^{k+1},\lambda^{k+1})$, 其中 $\mu\in(0,1)$ (通常取 $\mu=0.9$).

当 $\mu = 1$ 时, 方法 (5.7) 是可以举出不收敛的反例的. 我们用统一框架证明对称型乘子交替方向法 (5.7) 的收敛性, 也是把方法 (5.7) 拆解成预测-校正两部分. 对由 (5.7) 产生的 x^{k+1} 和 y^{k+1}, 我们按如下方式定义预测点 $\tilde{w}^k$:

$$\tilde{w}^k = \begin{pmatrix} \tilde{x}^k \\ \tilde{y}^k \\ \tilde{\lambda}^k \end{pmatrix} = \begin{pmatrix} x^{k+1} \\ y^{k+1} \\ \lambda^k - \beta(Ax^{k+1} + By^k - b) \end{pmatrix}. \tag{5.8}$$

利用 $\mathcal{L}_\beta^{[2]}(x, y, \lambda)$ 的表达式, 对称型的乘子交替方向法迭代公式可以表示成等价的

$$\left\{ \begin{array}{ll} \tilde{x}^k \in \operatorname{argmin}\{\theta_1(x) - x^T A^T \lambda^k + \frac{\beta}{2}\|Ax + By^k - b\|^2 \mid x \in \mathcal{X}\}, & \text{(5.9a)} \\ \lambda^{k+\frac{1}{2}} = \lambda^k - \mu\beta(A\tilde{x}^k + By^k - b), & \text{(5.9b)} \\ \tilde{y}^k \in \operatorname{argmin}\{\theta_2(y) - y^T B^T \lambda^{k+\frac{1}{2}} + \frac{\beta}{2}\|A\tilde{x}^k + By - b\|^2 \mid y \in \mathcal{Y}\}, & \text{(5.9c)} \\ \tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b). & \text{(5.9d)} \end{array} \right.$$

下面我们先找出 (5.9) 给出的 $\tilde{w}^k$ 在统一框架中形如 (1.2a) 的预测公式.

根据第一讲的定理 1, (5.9a) 的最优性条件是

$$\tilde{x}^k \in \mathcal{X}, \;\; \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{-A^T\lambda^k + \beta A^T(A\tilde{x}^k + By^k - b)\} \ge 0, \; \forall\, x \in \mathcal{X}.$$

利用 $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, 上式就是

$$\theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T \{\underline{-A^T\tilde{\lambda}^k}\} \geq 0, \quad \forall\, x \in \mathcal{X}. \tag{5.10a}$$

类似地, 根据第一讲的定理 1, (5.9c) 的最优性条件是

$$\tilde{y}^k \in \mathcal{Y}, \quad \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T \big\{-B^T\lambda^{k+\frac{1}{2}} + \beta B^T(A\tilde{x}^k + B\tilde{y}^k - b)\big\} \geq 0, \ \forall\, y \in \mathcal{Y}.$$

利用 $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, 我们有

$$\lambda^{k+\frac{1}{2}} = \lambda^k - \mu(\lambda^k - \tilde{\lambda}^k) = \tilde{\lambda}^k + (\mu - 1)(\tilde{\lambda}^k - \lambda^k),$$

和

$$\beta(A\tilde{x}^k + By^k - b) = -(\tilde{\lambda}^k - \lambda^k).$$

因此,

$$\begin{aligned}
&-B^T\lambda^{k+\frac{1}{2}} + \beta B^T(A\tilde{x}^k + B\tilde{y}^k - b) \\
&\quad = \quad -B^T[\tilde{\lambda}^k + (\mu - 1)(\tilde{\lambda}^k - \lambda^k)] + \beta B^T B(\tilde{y}^k - y^k) + \beta B^T(A\tilde{x}^k + By^k - b) \\
&\quad = \quad -B^T\tilde{\lambda}^k + (1 - \mu)B^T(\tilde{\lambda}^k - \lambda^k) + \beta B^T B(\tilde{y}^k - y^k) - B^T(\tilde{\lambda}^k - \lambda^k) \\
&\quad = \quad -B^T\tilde{\lambda}^k + \beta B^T B(\tilde{y}^k - y^k) - \mu B^T(\tilde{\lambda}^k - \lambda^k).
\end{aligned}$$

子问题 (5.9c) 的最优性条件是

$$\tilde{y}^k \in \mathcal{Y}, \ \ \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T\{\underline{-B^T\tilde{\lambda}^k} + \beta B^T B(\tilde{y}^k - y^k) \\ - \mu B^T(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \quad \forall y \in \mathcal{Y}. \tag{5.10b}$$

对于 (5.9d) 中定义的 $\tilde{\lambda}^k = \lambda^k - \beta(Ax^{k+1} + By^k - b)$, 由于

$$(A\tilde{x}^k + B\tilde{y}^k - b) - B(\tilde{y}^k - y^k) + (1/\beta)(\tilde{\lambda}^k - \lambda^k) = 0,$$

可以表示成

$$\tilde{\lambda}^k \in \Re^m, \quad (\lambda - \tilde{\lambda}^k)^T\{\underline{(A\tilde{x}^k + B\tilde{y}^k - b)} \\ - B(\tilde{y}^k - y^k) + (1/\beta)(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \quad \forall \lambda \in \Re^m. \tag{5.10c}$$

将 (5.10a), (5.10b) 和 (5.10c) 组合在一起并利用 (4.3) 中的记号, 我们有下面的引理.

引理 3 求解变分不等式 (4.3). 对给定的 v^k, 设 $\tilde{w}^k$ 是由 (5.9) 提供的, 则有

$$\tilde{w}^k \in \Omega, \ \ \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \ \ \forall w \in \Omega, \tag{5.11a}$$

其中

$$Q = \begin{pmatrix} \beta B^T B & -\mu B^T \\ -B & \frac{1}{\beta} I_m \end{pmatrix}. \tag{5.11b}$$

接着我们要导出形如 (1.2b) 的校正关系式. 利用 (5.8), 由 (5.7d) 给出的 λ^{k+1} 可以表示成

$$\begin{aligned}\lambda^{k+1} &= \lambda^{k+\frac{1}{2}} - \mu\big[-\beta B(y^k - \tilde{y}^k) + \beta(A\tilde{x}^k + By^k - b)\big] \\ &= [\lambda^k - \mu(\lambda^k - \tilde{\lambda}^k)] - \mu\big[-\beta B(y^k - \tilde{y}^k) + \beta(Ax^{k+1} + By^k - b)\big] \\ &= \lambda^k - \big[-\mu\beta B(y^k - \tilde{y}^k) + 2\mu(\lambda^k - \tilde{\lambda}^k)\big]. \end{aligned} \tag{5.12}$$

跟 $y^{k+1} = \tilde{y}^k$ 结合在一起, 就有

$$\begin{pmatrix} y^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} y^k \\ \lambda^k \end{pmatrix} - \begin{pmatrix} I & 0 \\ -\mu\beta B & 2\mu I_m \end{pmatrix} \begin{pmatrix} y^k - \tilde{y}^k \\ \lambda^k - \tilde{\lambda}^k \end{pmatrix}.$$

因此, 我们有下面的引理.

引理 4 求解变分不等式 (4.3). 对给定的 v^k, 设 w^{k+1} 由 (5.7) 提供. 那么对由 (5.8) 定义的 $\tilde{w}^k$, 我们有

$$v^{k+1} = v^k - M(v^k - \tilde{v}^k), \tag{5.13a}$$

其中

$$M = \begin{pmatrix} I & 0 \\ -\mu\beta B & 2\mu I_m \end{pmatrix}. \tag{5.13b}$$

我们已经把对称的 ADMM (5.7) 拆解成 (5.11) 的预测和 (5.13) 的校正. 剩下来的事情就是根

据统一框架中的收敛性条件 (1.3) 验证算法的收敛性. 对 (5.13b) 中的矩阵 M, 算出

$$M^{-1} = \begin{pmatrix} I & 0 \\ \frac{1}{2}\beta B & \frac{1}{2\mu} I_m \end{pmatrix}.$$

由 $H = QM^{-1}$ 得到

$$H = \begin{pmatrix} \beta B^T B & -\mu B^T \\ -B & \frac{1}{\beta} I_m \end{pmatrix} \begin{pmatrix} I & 0 \\ \frac{1}{2}\beta B & \frac{1}{2\mu} I_m \end{pmatrix} = \begin{pmatrix} (1-\frac{1}{2}\mu)\beta B^T B & -\frac{1}{2} B^T \\ -\frac{1}{2} B & \frac{1}{2\mu\beta} I_m \end{pmatrix}.$$

因此

$$H = \frac{1}{2} \begin{pmatrix} \sqrt{\beta} B^T & 0 \\ 0 & \sqrt{\frac{1}{\beta}} I \end{pmatrix} \begin{pmatrix} (2-\mu) I & -I \\ -I & \frac{1}{\mu} I \end{pmatrix} \begin{pmatrix} \sqrt{\beta} B & 0 \\ 0 & \sqrt{\frac{1}{\beta}} I \end{pmatrix} \tag{5.14}$$

注意到

$$\begin{pmatrix} (2-\mu) & -1 \\ -1 & \frac{1}{\mu} \end{pmatrix} = \begin{cases} \succ 0, & \mu \in (0,1); \\ \succeq 0, & \mu = 1. \end{cases}$$

所以, 对所有的 $\mu \in (0,1)$, 当 B 列满秩时矩阵 H 是对称正定的.

再看矩阵 $G = Q^T + Q - M^T HM$. 因为 $M^T HM = M^T Q$, 由

$$M^T Q \quad = \quad \begin{pmatrix} I & -\mu\beta B^T \\ 0 & 2\mu I_m \end{pmatrix} \begin{pmatrix} \beta B^T B & -\mu B^T \\ -B & \frac{1}{\beta} I_m \end{pmatrix} = \begin{pmatrix} (1+\mu)\beta B^T B & -2\mu B^T \\ -2\mu B & 2\mu \frac{1}{\beta} I_m \end{pmatrix}.$$

得到

$$\begin{aligned} G &= (Q^T + Q) - M^T HM \\ &= \begin{pmatrix} 2\beta B^T B & -(1+\mu) B^T \\ -(1+\mu) B & 2\frac{1}{\beta} I_m \end{pmatrix} - \begin{pmatrix} (1+\mu)\beta B^T B & -2\mu B^T \\ -2\mu B & 2\mu \frac{1}{\beta} I_m \end{pmatrix} \\ &= (1-\mu) \begin{pmatrix} \beta B^T B & -B^T \\ -B & \frac{2}{\beta} I_m \end{pmatrix}. \end{aligned} \tag{5.15}$$

同样, 对所有的 $\mu \in (0,1)$, 当 B 列满秩时矩阵 G 正定. 所以, 根据统一框架 (1.2a)-(1.2b), 方法是收敛的, 我们有下面的定理.

定理 5 求解变分不等式 (4.3). 对给定的 v^k, 设 w^{k+1} 由 (5.7) 提供. 我们有

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \|v^k - \tilde{v}^k\|_G^2, \quad \forall v^* \in \mathcal{V}^*,$$

其中 $\tilde{w}^k$ 由 (5.8) 定义,

$$H = \begin{pmatrix} (1 - \frac{1}{2}\mu)\beta B^T B & -\frac{1}{2}B^T \\ -\frac{1}{2}B & \frac{1}{2\mu\beta}I_m \end{pmatrix}$$

和

$$G = (1-\mu)\begin{pmatrix} \beta B^T B & -B^T \\ -B & \frac{2}{\beta}I_m \end{pmatrix}.$$

由于 $\mu \in (0,1)$, 矩阵 H 和 G 在 B 列满秩时都是正定的.

在矩阵 B 不一定列满秩的时候, 矩阵 H 和 G 半正定. 方法都具备定理 3 和定理 4 中的相关收敛速率性质.

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